A Matrix Model for QCD

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A reduced $3 \times N$-matrix model for $SU(N)$ gauge theory.

- Captures features of QCD like $\theta$-vacua.
- Gives mass gap in gluon spectrum.
- "Predicts" different phases of QCD – one seems Color Superconductivity.
- Confirms colored states are mixed. $\Rightarrow$ A new argument for color confinement.
On Gluons in Exact QCD

Call $A =$ Space of connections $A$.
We begin with wavefunctions $\Psi$:

$$\Psi : A \rightarrow \mathbb{C}$$

Gauss Law is:

$$D_i E_i \Psi = 0.$$ 

More precisely,

$$\left[ \text{Tr} \int \left( D_i \Lambda \right) E_i \right] \Psi \equiv G(\Lambda) \Psi = 0, \quad \Lambda(\vec{x}) \rightarrow 0, \text{ as } |\vec{x}| \rightarrow \infty$$

$$\Lambda(\vec{x}) = \left( \Lambda(\vec{x})^a : a = 1, 2, \ldots N \right).$$

The group it generates is

$$G_0^\infty = \{ g : \vec{x} \rightarrow SU(N), \ g(\vec{x}) \rightarrow e \text{ as } |\vec{x}| \rightarrow \infty \}.$$

So by Gauss law

$$\Psi : A / G_0^\infty \rightarrow \mathbb{C}.$$
The Global Group $SU(N)$

- It is generated by
  \[ \int Tr(D_i \mu) E_i = Q(\mu), \quad \mu \bigg|_{\infty} = \text{constant}. \]
  
  On the wave functions, only the global group $G / G_0^\infty = SU(N)$ acts nontrivially.

- We can use any $Q(\mu)$ as Charge:
  
  For any $\mu$ with fixed $\mu \bigg|_{\infty}$, $Q(\mu)$ all same on quantum states.

- Choice of constant $\mu$ for all $\vec{x}$ gives standard charge.

- The representations of $G / G^\infty = SU(N)$ give colored wave functions.
Gribov Problem

- It is impossible to gauge fix for group $G_0^\infty$:
  
  the bundle $G_0^\infty \to \mathcal{A} \to \mathcal{A}/G_0^\infty$ is twisted.

- **Locality**: Observables are
  - local
  - commute with Gauss law:

  $K = \text{observable, } [K, G(\Lambda)] = 0. \quad (1)$

- If $K$ is supported in the region $C$, (1) depends only on

  $\Lambda \big|_C = \mu \big|_C$ for some $\mu$.

- Conversely,

  $\mu \big|_C = \Lambda \big|_C$ for some $\Lambda$

  implies $[K, Q(\mu)] = 0$.

- Observables are color-singlets.
So state vectors can have global color, observables are color-singlets $\implies$ colored states are mixed (see later).

- We cannot gauge fix for bundle

$$\mathcal{A}/G_0^\infty \xrightarrow{SU(N)} \mathcal{A}/G.$$  

- Focus now on $SO(3)$ or $SU(2)$ for simplicity.
This feature is captured by connections

\[ A_i = \frac{i}{2} \text{Tr} \left( \frac{\tau^a}{2} g^{-1} \partial_i g \right) M_{ab} \tau^b \]

where \( M_{ab} \) is real, \( \vec{x} \rightarrow g(\vec{x}) \) is any one-to-one map from \( \mathbb{R}^3 \) to \( SU(2) \) with

\[ g|_{|\vec{x}|=\infty} = \text{constant map}. \]

Example: Skyrmion

\[ g(\vec{x}) = \cos \theta(r) + i \vec{\tau} \cdot \hat{x} \sin \theta(r), \quad \theta(0) = 0, \quad \theta(\infty) = \pi. \]
Spatial Rotations:
\[ g \rightarrow h^{-1}gh \]

or
\[
\text{Tr} \frac{\tau_a}{2} g^{-1} \partial_i g
\rightarrow
\text{Tr} \frac{h \tau_a h^{-1}}{2} g^{-1} \partial_i g
\]
\[ = \text{Tr} \frac{\tau_c}{2} (g^{-1} \partial_i g)) \vec{x}) R_{ca}(h) \]

or \( M \rightarrow RM, \quad R \in SO(3) = \text{Group of spatial rotations.} \)

SO(3) gauge transformation
\[
A_i dx^i = \Omega \rightarrow h' \Omega h'^{-1}
\]

or
\[
M_{ab} \tau_b \rightarrow h' M_{ab} \tau_b h'^{-1} = M_{ab} \tau_c S_{cb}(h')
\]

or \( M \rightarrow MS^T, \quad S \in SO(3) = \text{Gauge group SO(3).} \)
Curvature $F_{ij}$

Choose vector fields $L_i$ such that

$$L_i g = [g, \tau_i] = g\tau_i - \tau_i g$$

$$\implies A_i^a = iM_{ia}, \quad A_i = iM_{ia} \frac{\tau_a}{2}.$$ 

$$F_{ij} = i \left[ \epsilon_{ijk} M_{ka} + \epsilon_{abc} M_{ib} M_{jc} \right] \frac{\tau_a}{2}.$$ 

Note:
The bundle $S : M \to \mathbf{MS}^T$, $S \in \text{color } SO(3)$:

$SO(3) \to \text{Mat}_3(\mathbb{R}) \to \text{Mat}_3(\mathbb{R})/SO(3)$
is twisted $\iff$ the twist of QCD.
The potential $V(M)$

\[
V(M) = -\text{Tr}(F_{ij})^2
= \text{Tr}(M^T M) + 3\det(M) + \frac{1}{2} \left[ \left( \text{Tr}(M^T M) \right)^2 - \text{Tr} \left( (M^T M)^2 \right) \right].
\]

The matrix model for this potential has been studied:
Denjoe O’Connor and Rodrigo Delgadillo-Blando, Hoppe, Denjoe O’Connor and Filev,  

Including Kinetic Energy ($A_0 = 0$):
Kinetic Energy in QCD = $\text{Tr}(\dot{A}_i)^2$.

For us: $A^\alpha_i = i M_{i\alpha}$.

So, kinetic energy for us = $\text{Tr}(\dot{M}^T \dot{M})$

Or electric field for us = $E_{i\alpha} = -i \frac{\partial}{\partial M_{i\alpha}}$

Or Hamiltonian:

$$H = \frac{1}{R} \left[ -\frac{\partial^2}{(\partial M_{i\alpha})^2} + V(M) \right].$$
Hilbert space $\mathcal{H}$ and its inner product

- Wave functions $\psi : M \rightarrow \mathbb{C}$.
- For scalar product, choose metric

$$ds^2 = tr(dM^T)(dM)$$

invariant under $M \rightarrow RMS^T$.
- Gives volume form: $dV = \prod_{i\alpha} dM_{i\alpha}$.
- So

$$(\psi, \chi) = \int \prod (dM_{i\alpha}) \bar{\psi}(M) \chi(M).$$
Separation of variables

- Singular Value Decomposition:

\[ M = R \Delta S^T, \quad R, S \in SO(3) \]

\[ \Delta = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad a_1 \geq a_2 \geq a_3 \geq 0. \]

Focus on:

**Zero angular momentum, color singlet.**

Then

\[
dV = (a_1^2 - a_2^2)(a_2^2 - a_3^2)(a_1^2 - a_3^2) \prod (da_i) := \sqrt{g} \prod (da_i),
\]

\[
\sum \frac{\partial^2}{(\partial M_{i\alpha})^2} = \frac{\partial^2}{\partial a_i^2} + 2a_1 \left( \frac{1}{a_1^2 - a_2^2} + \frac{1}{a_1^2 - a_3^2} \right) + \text{cyclic.}
\]
For now, domain of Laplacian from smooth functions on $M$.
No special boundary conditions.
Integration over $a_1 \geq a_2 \geq a_3 \geq 0$.
Also,

$$V(M) = \frac{1}{2} a_i a_i - 2a_1 a_2 a_3 + \frac{1}{2} \left( a_1^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_1^2 \right).$$

Note: $V(M) \to \infty$ as $a_i \to \infty$ and $V(M) \geq 0$.
Hence expect a mass gap and only discrete levels.
The model has *naturally* defined **boundaries**.

They are like $r = 0$ of Laplacian on $\mathbb{R}^3$ in radial coordinates.

We can

- a) *either* treat $r = 0$ as a special point where say proton in hydrogen atom sits
- b) *or* treat it as coordinate singularity.

For **case a)**, a special boundary condition needed (See K.M. Case, Relativistic Hydrogen atom for Z>137).

**Boundaries here:** where volume form vanishes, that is $a_i = a_j$. 
Meaning of boundaries

From

\[ M = R \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} S^T, \]

if say \( a_1 = a_2 \),

\( M \rightarrow M, \) if \( R \rightarrow Rr(\theta), \) \( S \rightarrow Sr(\theta) \) where

\[ r(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Or

connection (holonomy) has higher symmetry.

Boundary conditions at these boundaries affect spectrum (work in progress).

Back to standard phase: Here we just work on \( \mathbb{R}^9 \).
Spectrum is gapped

Hamiltonian is

\[ H = -\frac{1}{\sqrt{g}} \partial_i \sqrt{g} \partial_i + V(M), \quad V(M) > 0. \]

\[
(\psi, H\psi) = \int_{a_1 \geq a_2 \geq a_3 \geq 0} \prod da_i \sqrt{g} \psi^\dagger H \psi \\
= \int \prod da_i \sqrt{g} (\partial_i \psi)^\dagger (\partial_i \psi) + \int \prod da_i \sqrt{g} \psi^\dagger V(M) \psi > 0
\]

and cannot be zero.

So spectrum is gapped.
Colored states are mixed

- Observables $K$ are color singlets.
- So consider, for example, a color octet vector in QCD:

$$|\cdot, \lambda\rangle, \quad \lambda = 1, 2 \cdots 8$$

Then,

$$P = \sum_\lambda |\cdot, \lambda\rangle \langle \cdot, \lambda|.$$

is a color singlet projector, hence an observable.

- So we can prepare state $|\cdot, P = 1\rangle$.
- But on observables $K$,

$$\langle \cdot, P = 1 | K | \cdot, P = 1 \rangle = \sum_\lambda \langle \cdot, \lambda | K | \cdot, \lambda \rangle$$

$$= \sum_\lambda \mu_\lambda \langle \cdot, \lambda | K | \cdot, \lambda \rangle \equiv \omega_\mu(K)$$

$$\sum \mu_\lambda = 1, \quad \mu_\lambda \geq 0.$$
Colored states are mixed

\[ \omega_\mu = \sum_\lambda \mu_\lambda |\cdot, \lambda\rangle \langle \cdot, \lambda| . \]

Thus since

\[ \langle \cdot, \lambda|K|\cdot, \lambda\rangle \]

is independent of \( \lambda \), \( K \) being a color singlet,

\[ \omega_\mu(K) \text{ is independent of } \mu. \]

So \( |\cdot, P\rangle \langle \cdot, P| \) is a mixed state.

But we cannot observe \( |\cdot, \lambda\rangle \langle \cdot, \lambda| \) which is not a color singlet.

So we cannot prepare a pure state!

Hence a generic colored state on observables is mixed.
Remarks on Confinement

- Unitary evolution cannot map pure state to a mixed state.
- So by unitary evolution a pure color singlet state cannot split into two colored and hence mixed states.
- Related to confinement?
- One can also prove that

\[ \sum \mu_\lambda \langle \cdot, \lambda | H | \cdot, \lambda \rangle = \infty \]  

(2) 

due to domain problems.

- For this conclusion (2), colored states must be mixed.