Regular monopoles in all dimensions

Tigran Tchrakian[‡]

[‡]School of Theoretical Physics – DIAS, 10 Burlington Road, Dublin 4, Ireland

June 5, 2009

Monpoles in all dimensions $\mathbb{R}^D$ can be constructed as solutions to Yang-Mills–Higgs equations with $SO(D)$ gauge connection.

Monopoles are distinguished from instantons in that the Yang-Mills connection of the monopole is asymptotically *half pure-gauge* rather than *pure-gauge* in the case of instanton.

This results in the r^{-1} decay of the connection of monopoles *vs.* the r^{-2} decay in the case of instantons, enabling the definition of Dirac–Yang monopoles.

Note that the curvature of an asymptotically *pure-gauge* vanishes on the spatial boundary, *i.e.*, excluding the possibilty of constructing Dirac–Yang monopoles!

The construction of the models supporting monopoles involves two steps, the definition of the Yang–Mills systems in higher dimensions, followed by dimensional reduction giving rise to the Higgs field. The plan of the talk is to present:

1. The hierarchy of Yang–Mills systems in all even dimensions
2. Dimensional reduction of Yang–Mills systems
3. The Dirac–Yang monopoles

1 The hierarchy of Yang–Mills systems

The fundamental building block is the curvature $2p-$form field strength defined interms of the 2-form Yang–Mills curvature $F(2)$

$$F(2p) = F(2) \wedge F(2) \wedge ... \wedge F(2) \ , \quad p - \text{times}$$

the p fold totally antisymmetrised product of the 2-form curvature.

The hierarchy of Yang–Mills systems consists of the Lagrangians constructed from the squares of these $2p-$forms

$$\mathcal{L}_p = \frac{1}{2(2p)!} \, \mathrm{Tr} \, F(2p)^2 \ , \qquad \text{defined on } \mathbb{R}^{2p} \, .$$

The most general Yang–Mills system is the superposition of these, with appropriate coupling strengths τ_p

$$\mathcal{L}_P = \sum_{p=1}^{P} \frac{\tau_p}{2(2p)!} \, \mathrm{Tr} \, F(2p)^2 \, ,$$

which can be defined in any dimension high enough to enable the antisymmetrisation, and with highest value of $p = P$, consistent with the dimensions.

The general Yang–Mills systems can be defined in both **even** and **odd** dimensions, **however**, they are of interest only in **even** dimensions $D = 2(p+q)$.

This is because topological lower bounds can be established only in **even** dimensions, which is necessary from the viewpoint of constructing topologically stable (instanton) solutions, and, of ”usefully” carrying out dimensional reduction.

The fundamental topological inequality is

$$\mathrm{Tr}[F(2p) - \kappa\ \ {}^{\star}F(2q)]^2 \geq 0\ ,$$

leading to

$$\mathrm{Tr}[F(2p)^2 + \kappa^2\ F(2q)^2] \geq 2\kappa\, \mathcal{C}_{p+q}\ ,$$

$\mathcal{C}_{p+q}$ is the $(p+q)$-th Chern-Pontryagin density.

When $q \neq p$ the first order equations that saturate this lower bound

$$F(2p)\ =\ \kappa\ {}^{\star}F(2q)$$

cannot be solved in closed form on $\mathbb{R}^{2(p+q)}$ and the (stable instanton) solutions to the second order Euler–Lagrange equations of this system can be evaluated **only** numerically.

On compact symmetric coset spaces however, *e.g.*, on CP^N, the lower bound can be saturated and closed form solutions constructed. In these cases the dimensionful constant κ plays the role of the "radius of curvature" of the space in question.

When $q = p$, *i.e.*, when $D = 4p$, then the dimensional parameter κ disappears and the first order equations

$$F(2p)\ =\ {}^{\star}F(2p)$$

saturate the lower bound and can be solved in closed form.

Note: For $p = 1$ this is the usual Yang–Mills system with the well known instanton solutions on $\mathbb{R}^4$. For $p \geq 2$ however, these first order equations are *overdetermined* and can be solved only when subjected to at least **axial symmetry**.

All solutions on $\mathbb{R}^{2(p+q)}$ described above are *instantons*, whose gauge connections are asymptotically *pure-gauge*

$$A \rightarrow g^{-1}\, dg$$

2 Dimensional reduction: Higgs models on $\mathbf{R}^D$

The above described Yang–Mills models support topologically stable solutions only on **even** space dimensions. To construct topologically stable solutions on **odd** space dimensions, one relies on exploiting the topological inequality used above, subjecting it to dimensional reduction.

It is the necessity of the topological inequality that restricts the dimensional reduction to take place from and even dimensional space $\mathbb{R}^D \times K^{2(p+q)-D}$ down to $\mathbb{R}^D$ with codimension $K^{2(p+q)-D}$ being a compact symmetric coset space. Here D can be bothe even or odd!

It is natural to take the bulk space to be $\mathbb{R}^D \times K^{4p-D}$, *i.e.*, $p = q$, to avoid the appearance of the dimensionful constant κ in the residual Yang-Mills–Higgs system on $\mathbb{R}^D$. In practice the appearance of the additional constant κ brings absolutely no advantage!

Moreover, it is sufficient for qualitative purposes to restrict to the codimension $K^{4p-D} = S^{4p-D}$ to be a sphere.

The topological inequality that is subjected to dimensional reduction is

$$\int_{\mathbf{R}_D \times S^{4p-D}} \mathrm{Tr}\mathcal{F}(2p)^2 \ \geq \int_{\mathbf{R}_D \times S^{4p-D}} \mathcal{C}_{2p} \ ,$$

where the gauge field $\mathcal{F}(2)$ in the bulk has a large enough gauge group so that the imposition of the symmetry on the codimension results in a broken gauge group of the residual model on $\mathbb{R}^D$.

After integrating over the compact coordinates of the sphere S^{4p-D} we end up with the residual energy on the left hand side.

The residual Lagrangian $\mathcal{L}[A,\Phi]$ is given both in terms of the residual gauge connection A_μ and a (gauge covariant) Higgs multiplet Φ.

The most important result in this procedure is that the density in the right hand side of the inequality is a **total divergence**, just as the Chern–Pontryagin density $\mathcal{C}_{2p}$ was also formally a **total divergence** before the imposition of symmetry!

$$\begin{aligned}\int_{\mathbf{R}^D} \mathcal{L}[A,\Phi] &\geq \int_{\mathbf{R}^D} \nabla\cdot\mathbf{\Omega}[A,\Phi] \\ &= \int_{\mathbf{\Sigma^{D-1}}} \mathbf{\Omega}[A,\Phi]\ .\end{aligned}$$

This is the new topological inequality that enebles the construction of monopoles in all dimensions.

Formalism of the dimensional reduction: (Formalism of A. S. Schwarz.)
Since both the Lagrangian density and the Chern–Pontryagin density are **rotational invariant**, and, **gauge invariant**, it is not necessary to display the explicit dependence of these densities on the angular coordinates on the S^{4p-D}.
It is sufficient therefore to evaluate these densities at a convenient fixed point of the sphere, namely at the North or the South pole.

The descent over **odd** codimensions differs essentially from that over **even** codimensions.

A salient feature of the above described mode of dimensional descent is that the Higgs multiplet can be expressed as an isovector of the residual gauge group $SO(D)$. In terms of elements of a representation of a group, it takes its values in the algebra of $SO(D+1)$.

In **odd** residual dimensions

- The gauge connection takes its values in $SO(D)$ and the Higgs field in $SO(D+1)$, whence the *chiral* representations of the even dimensional orthoganal group can be employed, with the connection in the $SO(D)$ subalgebra of $SO(D+1)$. (**Note:** for $D = 3$ the Higgs field is also in the $SU(2)$ algebra. This is a low dimensional accident.)
- The residual Chern-Simons desnsity, or monopole charge density, is **gauge invariant**!

In **even** residual dimensions

- The gauge connection takes its values in $SO(D)$ and the Higgs field in $SO(D+1)$, which is an odd dimensional orthognal group. As a result there is no possibility of the gauge connection taking its values in the *chiral* representation and must necessarily be in the full dirac representation. The Higgs multiplet is again an isovector of $SO(D+1)$.
- The residual Chern-Simons desnsity, or monopole charge density, is in this case **gauge variant**!

Apart from the different multiplet strucures in even and odd dimensions, the above distinctions are not very severe since most properties are representation independent – as long as **fermions are absent**.

We will display the formulas expressing the components of $(\mathcal{A}_\mu, \mathcal{A}_m)$, $\mu = 1, 2, \ldots, D$ being the indices on the residual space and $m = 1, 2, \ldots, (4p - D)$ those on the codimension, on the fixed point of the spehere S^{4p-D}. We restrict to the simplest cases $D = 3, 5$ and $D = 4$, both with $p = 2$.

Notation: In what follows spin matrix representations of the orthogonal groups is employed, with the following notation:

In terms of the gamma matrices the Γ_i, $i = 1, 2, \dots, M$ in $M-$dimensions, the representations of $SO(M)$ are

$$\Gamma_{ij} = -\frac{1}{4}\,\Gamma_{[i}\Gamma_{j]}$$

and when M is even, the *chiral* representations are

$$\Sigma_{ij}^{(\pm)} = P_\pm\,\Gamma_{ij} \quad , \quad P_\pm = \frac{1}{2}(1\!\!1 \pm \Gamma_{M+1})\,,$$

where Γ_{M+1} is the chiral matrix.

Odd dimensional residual space: $D = 3, 5$

The calculus of descent to odd D is considerably simpler, so exhibit these two cases together. On the fixed point, *i.e.*, with no explicit dependence on the angles on S^5 and S^3,

$$\begin{aligned} \mathcal{A}_\mu &= A_\mu \otimes 1\!\!1 \\ \mathcal{A}_m &= \Phi \otimes \Gamma_m \end{aligned}$$

where Γ_m are the gamma matrices in 5 and 3 dimensions, respectively. The Higgs field can be expressed as

$$\Phi = \phi_\mu\,\Sigma_{\mu,D+1}$$

and both A_μ and Φ take their values in the chiral representation of the algebra of $SO(D+1)$ $(\Sigma_{\mu\nu}, \Sigma_{\mu,D+1})$, the former in the $SO(D)$ subalgebra. Note that both A_μ and Φ does no have to be traceless, thus there is an Abelian gauge field present.

The resulting components of $\mathcal{F}_{ij} = (\mathcal{F}_{\mu\nu}, \mathcal{F}_{\mu m}, \mathcal{F}_{mn})$ are

$$
\begin{aligned}
\mathcal{F}_{\mu\nu} &= F_{\mu\nu} \otimes \mathbb{1} \\
\mathcal{F}_{\mu m} &= D_\mu \Phi \otimes \Gamma_m \quad , \quad D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi] \\
\mathcal{F}_{mn} &= S \otimes \Gamma_{mn} \quad , \quad S = \eta^2 + \Phi^2
\end{aligned}
$$

where η is the inverse of the radius of the codimension sphere and plays the role of the Higgs VEV.

Even dimensional residual space: $D = 3, 5$

In this case the solution of the symmetry conditions is more general,

$$
\begin{aligned}
\mathcal{A}_\mu &= A_\mu^{(+)} \otimes P_+ + A_\mu^{(-)} \otimes P_- \\
\mathcal{A}_m &= \varphi \otimes P_+ \Gamma_m - \varphi^\dagger \otimes P_- \Gamma_m
\end{aligned}
$$

resulting in

$$
\begin{aligned}
\mathcal{F}_{\mu\nu} &= F_{\mu\nu}^{(+)} \otimes P_+ + F_{\mu\nu}^{(-)} \otimes P_- \\
\mathcal{F}_{\mu m} &= D_\mu \varphi \otimes P_+ \Gamma_m - D_\mu \varphi^\dagger \otimes P_- \Gamma_m \\
\mathcal{F}_{mn} &= S^{(+)} \otimes P_+ \Gamma_{mn} + S^{(-)} \otimes P_- \Gamma_{mn}
\end{aligned}
$$

with

$$
D_\mu \varphi = \partial_\mu \varphi + A_\mu^{(+)} \varphi - \varphi A_\mu^{(-)}
$$

and

$$
S^{(+)} = \varphi \varphi^\dagger - \eta^2 \quad , \quad S^{(+)} = \varphi^\dagger \varphi - \eta^2 \, .
$$

These expressions can be simplified by adopting the following notation

$$
A_\mu = \begin{bmatrix} A_\mu^{(+)} & 0 \\ 0 & A_\mu^{(-)} \end{bmatrix} \quad , \quad \Phi = \begin{bmatrix} 0 & \varphi \\ -\varphi^\dagger & 0 \end{bmatrix} \quad , \quad S = \begin{bmatrix} 0 & S^{(+)} \\ S^{(-)} & 0 \end{bmatrix} \, ,
$$

with

$$
F_{\mu\nu} = \begin{bmatrix} F_{\mu\nu}^{(+)} & 0 \\ 0 & F_{\mu\nu}^{(-)} \end{bmatrix} \quad , \quad D_\mu \Phi = \begin{bmatrix} 0 & D_\mu \varphi \\ -D_\mu \varphi^\dagger & 0 \end{bmatrix} \, ,
$$

and now with the unifyinging notation for both even and odd D,

$$D_\mu\Phi = \partial_\mu\Phi + [A_\mu, \Phi]\,.$$

Finally we substitute these field strengthe into the two sides of the new topological inequality on $\mathbb{R}^D$, and express the residual Lagrange densities $\mathcal{L}$, and more remarkably, the monopole (residual Chern–Pontryagin) densities ϱ.

For the $D = 3$ Higgs model these are

$$\begin{aligned}
\mathcal{L}_\ni &= \mathrm{Tr}\Big(\{F_{[\mu\nu}, D_{\rho]}\Phi\}^2 - 6(\{S, F_{\mu\nu}\} + [D_\mu\Phi, D_\nu\Phi])^2\,, \\
&\qquad\qquad -27\,\{S, D_\nu\Phi\}^2 + 54\,S^4\Big) \\
\varrho^{(3)} &= 6\varepsilon_{\mu\nu\rho}\mathrm{Tr}\Big(S^2\{F_{[\mu\nu}, D_{\rho]}\Phi\} + 3\{S, D_\rho\Phi\}(\{S, F_{\mu\nu}\} + [D_\mu\Phi, D_\nu\Phi])\Big) \\
&= 36\varepsilon_{\mu\nu\rho}\partial_\rho\mathrm{Tr}\Big[\Phi(3\eta^4 + 2\eta^2\Phi^2 + \frac{3}{5}\Phi^4)F_{\mu\nu} \\
&\qquad -2\eta^2\Phi D_\mu\Phi D_\nu\Phi - \frac{2}{5}\Phi^2(2\Phi D_\mu\Phi - D_\mu\Phi\Phi)D_\nu\Phi\Big]\,.
\end{aligned}$$

The second line in ϱ is the **total divergence** which defines the Chern–Simons density of this model. Its leading term is the magnetic field in this example, which is a **gauge covariant** quantity as expected.

This model is descende from the $p = 2$ Yang–Mills system and is only one of infinitely many $3-$dimensional generalisations of the $3-$dimensional Georgi-Glashow model in the BPS limit, descended fro all $p-$Yang–Mills. It has the *curious* feature that it supports *mutually attracting like charged monopoles.*

For the $D = 4$ Higgs model desceded from the $p = 2$ Yang–Mills system is

$$\mathcal{L}_4 = \mathrm{Tr}[F_{\mu\nu\rho\sigma}^2 + 4\lambda_1\{F_{[\mu\nu}, D_{\rho]}\Phi\}^2 - 18\lambda_2(\{(\eta^2+\Phi^2), F_{\mu\nu}\} - [D_{[\mu}\Phi, D_{\nu]}\Phi])^2$$
$$-54\lambda_3\{(\eta^2+\Phi^2), D_\mu\Phi\}^2 + 54\lambda_4(\eta^2+\Phi^2)^4] \qquad (1)$$

$$\begin{aligned}\varrho^{(4)} &= \frac{1}{64\pi^2}\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,\Gamma_5\Big[S^2 F_{\mu\nu\rho\sigma} + 4\{S, D_\mu\Phi\}\{F_{[\mu\nu}, D_{\rho]}\Phi\} \\ &+ 3\left(\{S, F_{\mu\nu}\} + [D_\mu\Phi, D_\nu\Phi]\right)\left(\{S, F_{\rho\sigma}\} + [D_\rho\Phi, D_\sigma\Phi]\right)\Big] \\ &= \frac{1}{64\pi^2}\,\varepsilon_{\mu\nu\rho\sigma}\,\partial_\mu \mathrm{Tr}\,\Gamma_5\Big[\eta^4 A_\nu\left(F_{\rho\sigma} - \frac{2}{3}A_k A_l\right) + \frac{1}{6}\eta^2\Phi\{F_{[\nu\rho}, D_{\sigma]}\Phi\} \\ &\qquad + \frac{1}{6}\Phi\,\left(\{S, F_{\rho\sigma}\} + [D_\rho\Phi, D_\sigma\Phi]\right) D_\nu\Phi\Big]\end{aligned}$$

Note that again the last expression of ϱ again displays the Chern–Simons density, which now is a **gauge variant** quantity. Note also the presence of the chiral matrix Γ_5 in ϱ, thus rendering the Chern–Simons density different from the usual $SU(2)$ one in 4 dimensions, but rather one for the full $SO(4)$ group.

3 Dirac-Yang monopoles

The Dirac-Yang monopoles arise as the gauge transformed asymptotic fields of the monoples. The Higgs field is gauged to a trivial constant, while the gauge connection now takes its values in $SO(D-1)$ and developes a line singularity along the x_D-axis.

The asymptotic fields of the $D-$dimensional monopole are

$$A_i^{(\pm)} = \frac{1}{r}\Sigma_{ij}^{(\pm)}\,\hat{x}_j \quad , \quad \Phi = \hat{x}_i\,\Sigma_{i,D+1}^{(\pm)} \quad , \quad \text{for odd } \; D$$

$$A_i^{(\pm)} = \frac{1}{r}\Gamma_{ij}\,\hat{x}_j \quad , \quad \Phi = \hat{x}_i\,\Gamma_{i,D+1} \quad , \quad \text{for even } \; D\,.$$

The Dirac–Yang monopoles result from the action of the following $SO(D)$ gauge group element

$$g_{\pm} = \frac{(1\pm\cos\theta_1)\mathbb{1} \pm \Gamma_D\Gamma_\alpha\hat{x}_\alpha\sin\theta_1}{\sqrt{2(1\pm\cos\theta_1)}}\,,$$

having parametrised the $\mathbb{R}^D$ coordinate $x_i = (x_\alpha, x_D)$ in terms of the radial variable r and the polar angles

$$(\theta_1, \theta_2, .., \theta_{D-2}, \varphi)$$

resulting in the Dirac–Yang connection

$$\hat{A}_\alpha^{(\pm)} = \frac{1}{r(1\pm\cos\theta_1)}\Sigma_{\alpha\beta}\hat{x}_\beta \quad , \quad \hat{A}_D^{(\pm)} = 0 \quad , \quad \text{for odd } D$$

$$\hat{A}_\alpha^{(\pm)} = \frac{1}{r(1\pm\cos\theta_1)}\Gamma_{\alpha\beta}\hat{x}_\beta \quad , \quad \hat{A}_D^{(\pm)} = 0 \quad , \quad \text{for even } D\,,$$

We display the resulting for the corresponding curvature field strengths

$$\hat{F}^{(\pm)}_{\alpha\beta} = -\frac{1}{r^2}\left[\Gamma_{\alpha\beta} + \frac{1}{(1 \pm \cos\theta_1)}\,\hat{x}_{[\alpha}\,\Gamma_{\beta]\gamma}\hat{x}_{[\gamma}\right] \tag{2}$$

$$\hat{F}^{(\pm)}_{\alpha D} = \pm\frac{1}{r^2}\,\Gamma_{\alpha\gamma}\hat{x}_{\gamma}\,, \tag{3}$$

for the even D case, the odd D case given by replacing the matrices Γ by the matrices Σ.