New Results on Semiclassical Corrections to General Relativity and to Photon Propagation

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Contents:

- Semiclassical approach to Quantum Gravity
- Quantum corrections from massive fields.
- Conformal anomaly, induced action and their ambiguities.
- Quantum corrections in the photon sector and MA.

Why semiclassical approach?

General Relativity (GR) is a complete theory of classical gravitational phenomena. However its perfectness is spoiled by singularities which are present in the most important solutions such as Schwarzschild and cosmological ones.

At very short distances / great cuvature, gravity should be described by another theory, more general than the GR.

Indeed, we expect a new theory being close to GR in the low energy / weak field limit.

The most probable origin of assumed deviation from GR are quantum effects.

The expected scale of quantum effects is associated to Planck units,

- length l_P, time t_P and mass M_P,

$$l_P = G^{1/2}\,\hbar^{1/2}\,c^{-3/2} \approx 1.4 \cdot 10^{-33}\,cm;$$

$$t_P = G^{1/2}\,\hbar^{1/2}\,c^{-5/2} \approx 0.7 \cdot 10^{-43}\,sec;$$

$$M_P = G^{-1/2}\,\hbar^{1/2}\,c^{1/2} \approx 0.2 \cdot 10^{-5} g \approx 10^{19}\,GeV\,.$$

We assume the existence of the fundamental units does indicate some fundamental physics at the M_P scale.

It may be Quantum Gravity, String theory, ...

We do not know what it really is.

So, what is certain?

Quantum Field Theory (QFT) and curved space definitely are.

Therefore the first step should be to consider **QFT of matter fields in curved space.**

Different from quantum theory of metric,
QFT of matter in curved space is well defined.

This theory is renormalizable and
is free of conceptual problems.

However, deriving the most relevant observables is,
in general, an unsolved problem.

Formulating QFT in curved space.

The first step: choose the action for gravity and matter.
We assume the vacuum part of the action includes **Einstein-Hilbert term** with a cosmological constant **(CC)**.

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\,\{\,R + 2\Lambda\,\}\,.$$

Why do we need other terms?

Standard criteria for the action of external metric:

• Locality. • Renormalizability. • Simplicity.

$$S_{vac} = S_{EH} + S_{HD}\,,$$

$$S_{HD} = \int d^4x\sqrt{-g}\,\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\}\,,$$

where $C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + (1/3)\,R^2\,, \qquad E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2\,.$

Quantum corrections to the vacuum action.

$$e^{i\Gamma[g_{\mu\nu}]} = \int \mathcal{D}\phi \, e^{iS[\phi;\, g_{\mu\nu}]}\,.$$

$\Gamma[g_{\mu\nu}]$ **is the Effective Action of gravity, which is a well-defined diffeomorphism invariant quantity**.

There are no gravitational anomalies in $4d$.
As a consequence, $\Gamma[g_{\mu\nu}]$ **can not include odd powers of metric derivatives.**

An odd-power behavior in the gravitational solutions, would be a direct indication to a kind of a "new physics", e.g. scalar field.

It can not be a vacuum quantum effect on purely metric background!

Particular application to cosmology.

Quantum corrections to CC in the late universe, without scalar fields, may only start from H^2

$$\rho_\Lambda(H) \;=\; \rho_\Lambda(H_0) + \nu M^2 \cdot \left(H^2 - H_0^2\right)\,, \qquad (*)$$

where H_0 is the present-day Hubble parameter.

Quantum physics below the GUT scale is irrelevant in the case of ρ_Λ .

Cosmological models based on eq. (*) were developed in

I.Sh., J. Solà, C. España-Bonet, P. Ruiz-Lapuente, Ph.L. B574 (2003) 149; JCAP 02 (2004);
I.Sh., J.Solà, H.Stefancic, JCAP 0501 (2005).
J.Fabris, I.Sh., J.Sola, JCAP (2007).
Partially based on the ideas of:
I.Sh., J.Solà, JHEP 02 (2002);
A.Babic, B.Guberina, R.Horvat, H.Stefancic, PRD 65 (2002).

Key question:

What could be the Effective Action (EA) of vacuum representing the **quantum** $\mathcal{O}(H^2)$ **correction to the cosmological constant?**

I.Sh., J. Solà, *On the possible running of the cosmological "constant"* Submitted to PLB, Dec. 2008. Extended version: arXiv:0808.0315.

It is obious that the corresponding **local** term can be nothing else but

$$\int_x R = \int d^4x \sqrt{-g(x)}\, R\,.$$

However there may be nonlocal terms too. Of course, these are strong restrictions on their form. E.g. the Polyakov-like 'massive' term

$$\int_x \int_y R \frac{m^2}{\Box + m^2} R$$

behaves as $\mathcal{O}(H^6)$ at far IR and hence is negligible.

At the same time we can still have an infinite amount of terms which are not related to the power series in curvature tensor. These terms are not ruled out at the present stage of our knowledge of the subject.

The EA of gravity $\Gamma[g_{\mu\nu}]$ admits a loop expansion

$$\Gamma[g_{\mu\nu}] = S_{vac}[g_{\mu\nu}] + \bar{\Gamma}^{(1)} + \bar{\Gamma}^{(2)} + \bar{\Gamma}^{(3)} + ... ,$$

The simplest (and usually most important) 1-loop corrections are given by

$$\bar{\Gamma}^{(1)} = \frac{i}{2}\,\mathrm{sTr}\ \mathrm{Log}\,\left(\hat{H}\right),$$

where

$$\hat{H} \,=\, \hat{H}(x,y) \,=\, \frac{1}{2}\,\frac{\delta^2\, S[\phi,\, g_{\mu\nu}]}{\delta\phi(x)\,\delta\phi(y)}\bigg|_{\phi=0}\,.$$

The main known methods to calculate quantum corrections are the following:

- **Feynman diagrams for** $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$.

R. Utiyama & B.S. De Witt, 1962 .
A.A. Starobinsky & Ya.B. Zeldovich, 1971 .
......

The shortcomings of this method are the lack of explicit covariance, difficulties in practical calculations and interpreting the results.

Definitely, this method is not approproate for calculating the quantum corrections to the vacuum energy.
I.Sh., J. Solà, (2002-2009)
I.Sh., Class.Quant.Grav. 25 (2008)(Topical review). gr-qc/0801.0216.
.

- **Schwinger-DeWitt (Sch-DW) expansion is the most useful method for deriving divergences and related quantities.**

$$\frac{i}{2}\,\mathrm{Tr}\,\ln\hat{H} = -\frac{i}{2}\,\mathrm{Tr}\int_0^\infty \frac{ds}{s}\,e^{-is\,\hat{H}}\,.$$

where

$$e^{-is\,\hat{H}} = \hat{U}_0(x,x'\,;s)\,\sum_{k=0}^{\infty}(is)^k\,\hat{a}_k(x,x')\,,$$

$\hat{a}_k(x,x')$ **are Sch-DW coefficients.**

• Quantum corrections in massive case.

The non-local part of the vacuum effective action of quantum fields would have numerous applications, e.g.,

Starobinsky model (massless fields effects)
Fischetti, Hartle and Hu (1978);
Starobinsky, (1980-1983) ;
Mukhanov, Chibisov, (1982) ;
Anderson, Vilenkin, ... (1983-1986)
Hawking, Hertog and Real, (2001) .

Modified Starobinsky model
(Fabris, Pelinson, Sola, Sh., ...) .

Possible running of CC at low energies
(Sola, Sh., Guberina et al, ...) .

An important aspect is **Decoupling of Heavy Massive Fields.**

The effective approach implies low-energy phenomena being independent on (sometimes unknown) fundamental physics.

One example is low-energy QCD, where the Chiral Perturbation Theory helps to fit both lattice simulations and experiment.

Do we have the decoupling for gravity?

At classical level decoupling means that a heavy field almost doesn't propagate at low energies

$$\frac{1}{k^2 + M^2} \approx \frac{1}{M^2} + \mathcal{O}\left(\frac{k^2}{M^4}\right), \qquad k^2 \ll M^2 .$$

Decoupling theorem explains similar phenomenon at quantum level.

Consider massive scalar field,

$$S_s = \frac{1}{2}\int d^4x\, g^{1/2}\,\Big\{(\nabla\varphi)^2 + m^2\varphi^2 + \xi R\varphi^2\Big\}\,.$$

Euclidean Effective Action is

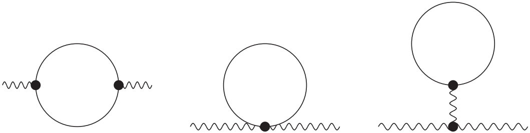
$$\Gamma[g_{\mu\nu}] = -\frac{1}{2}\,\mathrm{Tr}\,\ln\big(-\nabla^2 + m^2 + \xi R\big)\,.$$

Weak point: No covariant version of a mass-dependent renormalization scheme.

We can perform calculations only for the linearized gravity on the flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,.$$

Corrections to the graviton propagator:



The polarization operator must be compared to the tensor structure of the Lagrangians

$$L_{HE} = -\frac{1}{16\pi G}\,(R + 2\Lambda) \qquad \text{and}$$

$$L_{HD} = a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2 \,.$$

$$L_{HE} = -\frac{1}{16\pi G}\,(R + 2\Lambda) \qquad \text{and}$$

$$L_{HD} = a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2 \,.$$

For the formfactors we find, e.g.

$$k_\Lambda \,=\, \frac{3m^4}{8\,(4\pi)^2}\,, \qquad k_R \,=\, \frac{m^2}{2\,(4\pi)^2}\,\tilde{\xi}\,,$$

$$k_1(a) \,=\, \frac{8\,Y}{15\,a^4} \,+\, \frac{2}{45\,a^2} \,+\, \frac{1}{150}\,,$$

where $\tilde{\xi} = \xi - 1/6$,

$$Y = 1 + \frac{1}{a}\ln\left|\frac{2-a}{2+a}\right|\,, \quad a^2 = \frac{4\Box}{4m^2 - \Box}.$$

The result for the formfactors confirmed using covariant $\mathcal{O}(R^2)$ **heat kernel solution**

Avramidi, *Sov.J.Nucl.Phys.* 49 (1989);
Barvinsky, Vilkovisky, *Nucl.Phys*. B282 (1990),

(properly generalized for a massive case).

Similar expressions for the formfactors of C^2 and R^2 terms were obtained for massive **fermions and vectors.**
E. Gorbar & I.Sh. JHEP 06-2003.

In the theory with SSB
E. Gorbar & I.Sh. JHEP 02-2004.

In the $\lambda\phi^4$ theory,
E. Gorbar, G. Berredo-Peixoto & I.Sh. Class. Quant. Grav. **21** (2004).

In the curved-space QED
B. Gonçalves, G. de Berredo-Peixoto & I. Sh., 0904.4171 (hep-th).

How do we define RG in curved space?

Remember we are dealing with the theory of $h_{\mu\nu}$ **in flat space.**
Then RG scaling is the momentum scaling

$$p^2 \to e^{2t} p^2.$$

In the **mass-dependent scheme**

$$\beta_\lambda = -2p^2 \frac{\partial \lambda}{\partial p^2},$$

where we identify $p^2 = -\Box$. For the Weyl term:

$$\beta_1 = -\frac{1}{(4\pi)^2} \left(\frac{1}{18a^2} - \frac{1}{180} - \frac{a^2-4}{6a^4} Y \right).$$

Then $\beta_1^{UV} = -\frac{1}{120\,(4\pi)^2} + \mathcal{O}\left(\frac{m^2}{p^2}\right) = \beta_1^{\overline{MS}} + \mathcal{O}\left(\frac{m^2}{p^2}\right),$

$$\beta_1^{IR} = -\frac{1}{1680\,(4\pi)^2} \cdot \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right),$$

Appelquist & Carazzone Theorem for **gravity!**

Big Problem: In the **perturbative** (in $h_{\mu\nu}$) approach we do not observe the RG running of G and Λ.

Is it true that physical $\beta_\Lambda = \beta_{1/G} = 0$ **?**

No. Physical β_Λ and $\beta_{1/G}$ should tend to the corresponding $\beta^{\overline{MS}}$-functions in the UV limit. And $\beta_G^{\overline{MS}} \neq 0$, $\beta_\Lambda^{\overline{MS}} \neq 0$.

It is definitely true that the linearized gravity approach is not appropriate for the CC and Einstein terms.

What would happen if decoupling for CC is the standard one?

I.Sh., J.Solà, JHEP 02 (2002);
A.Babic, B.Guberina, R.Horvat, H.Stefancic, PRD 65 (2002).

$$\beta_\Lambda(IR) \sim \frac{\mu^2}{m^2}\, \beta_\Lambda(UV)\,.$$

- **But** $\beta_\Lambda^{(UV)}(m) \propto m^4$, also all massive particles provide additive contributions to β_Λ;

- We identify $\mu \sim H$ in cosmology.

Then, we arrive at

$$\beta_\Lambda \,=\, \frac{1}{(4\pi)^2}\,\sigma\, M^2\, H^2\,,$$

where M is unknown **mass parameter** and $\sigma = \pm 1$ depending on whether fermions or bosons dominate **at the highest energies.**

It is easy to see that the RG with

$$\beta_\Lambda = \frac{1}{(4\pi)^2} \, \sigma \, M^2 \, H^2 \, ,$$

leads to

$$\rho_\Lambda(H) = \rho_\Lambda(H_0) + \nu M^2 \cdot \left(H^2 - H_0^2\right) \, , \qquad (*)$$

with important cosmological implications.

In order to check whether the corresponding quantum corrections really take place, one needs a method which is not based on the power series expansion in curvature tensor.

Such method exists, but unfortunately it works only for massless conformal case.

Conformal anomaly.

Consider massless conformal fields.

The theory includes $g_{\mu\nu}$ and matter fields Φ.

k_Φ is the conformal weight of the field.

The local conformal Noether identity is

$$\left[-2\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\,\Phi\,\frac{\delta}{\delta\Phi}\right]\,S(g_{\mu\nu},\,\Phi)\,=\,0$$

It produces $T^\mu_\mu = 0$ for the vacuum (on shell for matter fields)

$$-\,\frac{2}{\sqrt{g}}\,g_{\mu\nu}\,\frac{\delta S_{vac}(g_{\mu\nu})}{\delta g_{\mu\nu}}\,=\,T^\mu_\mu\,=\,0\,.$$

At quantum level $S_{vac}(g_{\mu\nu})$ is replaced by the EA, $\Gamma_{vac}(g_{\mu\nu})$.

For free field only 1-loop order is relevant.

$$\Gamma_{div} \,=\, \frac{1}{\varepsilon} \int d^4x\sqrt{g}\,\{\beta_1 C^2 + \beta_2 E + \beta_3 \Box R\} \,.$$

For the **global** conformal symmetry the renormalization group tells

$$< T^\mu_\mu > = \,\{\beta_1 C^2 + \beta_2 E + a' \Box R\} \,,$$

where $a' = \beta_3$. **In the local conformal case a' is ambiguous.**

Duff, Class.Quant.Grav. (1994);

Asorey, Gorbar & I.Sh., Class.Quant.Grav. **21** (2003).

Consider the theory including massless fields,

N_0 scalars, $N_{1/2}$ spinors, N_1 vectors.

Conformal Anomaly

$$T = < T^\mu_\mu > = -(wC^2 + bE + c\,\Box R + \beta F^2_{\mu\nu})\,,$$

where

$$\begin{pmatrix} w \\ -b \\ c \end{pmatrix} = \frac{1}{360(4\pi)^2} \begin{pmatrix} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{pmatrix}$$

Ambiguity of local anomalous terms

Asorey, Gorbar & I.Sh., CQG **21** (2003).

The ambiguity of local anomalous term $\int\sqrt{-g}R^2$ in the effective action and the corresponding term $\Box R$ in the anomaly can be observed either in **dimensional or covariant Pauli-Villars regularizations.**

Consider the dimensional regularization.

As we have already mentioned, the counterterm $\int\sqrt{-g}\Box R$ doesn't contribute to anomalous violation of local conformal symmetry.

According to Duff (1977), the anomaly comes from the

$\int\sqrt{-g}C^2(d)$-type counterterm.

$$-\frac{2}{\sqrt{-g}}\; g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\,\int d^n x\sqrt{-g}\,C^2(d)\Big|_{d=4,\,n\to 4} \;=\; C^2-\frac{2}{3}\Box R\,.$$

However, the requirements of finiteness of renormalized effective and the locality leaves the freedom to choose the parameter d. If we take

$$d = n + \gamma \cdot [n - 4]\,,$$

where γ is an arbitrary parameter, we meet $a' \sim \gamma$.

The coefficient α' is arbitrary. Changing α' is equivalent to adding a $\int R^2$-term to the classical action.

Why we are allowed to add a $\int R^2$-term?

Because it would be a component of the action of **external fields**, which doesn't influence the dynamics of **quantum** fields.

In order to fix the arbitrariness in $\int R^2$ -term, one has to do the following:

- Introduce it into the classical action.
- Calculate quantum correction.
- Fix overall $\int R^2$-term by the renormalization condition.

Consider massive scalar field (Gorbar & I.Sh.) JHEP, 2003.

$$L = (1/2)\,\Big\{(\nabla\varphi)^2 + m^2\varphi^2 + \big(\tilde{\xi} + 1/6\big)\,R\varphi^2\Big\}\,.$$

In the second order in curvatures

$$\bar{\Gamma}^{(1)} = \int \frac{d^4x\sqrt{g}}{2(4\pi)^2}\,R\,\Big\{\frac{\tilde{\xi}^2}{2\epsilon} + Y\tilde{\xi}^2 + \frac{\tilde{\xi}\,Y(4-a^2)}{6a^2}$$

$$+\frac{\tilde{\xi}}{18} + \frac{Y\left(16-8a^2+a^4\right)}{144a^4} + \frac{20-7a^2}{2160\,a^2}\Big\}\,R + \dots,$$

$$Y = 1 + \frac{1}{a}\ln\left|\frac{2-a}{2+a}\right| \quad \text{and} \quad a^2 = \frac{4\Box}{4m^2-\Box}\,.$$

$$m=0,\;\; \xi = 1/6 \quad \Rightarrow \quad -\frac{1}{12\cdot 180(4\pi)^2}\int d^4x\, g^{1/2}\, R^2\,,$$

fitting perfectly with the conformal anomaly obtained by point-splitting (Christensen, 78), ζ-reg. (Cristley & Dowker, 76; Hawking, 77), etc.

$$< T^\mu_\mu >= \frac{1}{180(4\pi)^2}\,\Box R + \dots\,.$$

In the covariant Pauli-Villars regularization one has to introduce a set of “regulator” fields. E.g., in case of a massless conformal scalar φ we have to start from the action

$$S_{\text{reg}} = \sum_{i=0}^{N} \int d^4x \sqrt{g} \left\{ (\nabla\varphi_i)^2 + (\xi_i R + m_i^2)\varphi_i^2 \right\}.$$

The physical scalar field $\varphi \equiv \varphi_0$ is conformal $\xi = 1/6$, $m_0 = 0$ and bosonic $s_0 = 1$, while PV regulators φ_i are massive $m_i = \mu_i M$ and can be bosonic $s_i = 1$ or fermionic $s_i = -2$.

The UV limit $M \to \infty$ produces the vacuum Eff. Action. The calculation is based on our result for the EA of the massive scalar. We assume that the Pauli-Villars regulators may have conformal $\xi_i = 1/6$ or non-conformal couplings $\xi_i \neq 1/6$.

Direct calculation shows that the $\int R^2$-term depends on the choice of ξ_i and hence is arbitrary
Asorey, Gorbar & I.Sh., CQG **21** (2003).

Even stronger arbitrariness

M.Asorey, G. de Berredo-Peixoto, I.Sh, PRD-2006.

Consider interacting conformal scalar theory

$$S = \int d^4x\sqrt{g}\,\left\{\,\frac{1}{2}\,\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{12}\,R\phi^2 - \frac{\lambda}{24}\phi^4\,\right\}\,.$$

The Noether identity $\mathcal{T} \,=\, \frac{1}{\sqrt{g}}\left(\phi\,\frac{\delta S}{\delta\phi} - 2\,g_{\mu\nu}\,\frac{\delta S}{\delta g_{\mu\nu}}\right) \,=\, 0\,.$

At quantum level it is indeed violated $<\mathcal{T}>\neq 0$.

The ambigous part of anomaly $<\mathcal{T}>= \alpha_2'\Box\phi^2 + ...$ **h because the corresponding term in the EA is local**

$$\Gamma_{ind} \,=\, \frac{\alpha'}{6}\,\int d^4x\sqrt{g}\;R\phi^2\,+\,...$$

Changing $R\varphi^2$-term in the classical action implies an essential change in the dynamics of quantum fields !!

Most recent result of our group

Quantum correction to photon sector

Classical action:

$$S_{em} = -\frac{1}{4}\int d^4x\sqrt{g}\;F_{\mu\nu}\,F^{\mu\nu}$$

possesses local conformal invariance,

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}\,e^{2\sigma(x)}\,, \quad A_\mu \to A'_\mu = A_\mu\,,$$

Why it is important that the local conformal symmetry is violated at quantum level?

1) **Broken conformal symmetry changes the equation of state (EOS) of radiation. Modified EOS for the radiation may lead to interesting new cosmological models.**
Modified EOS changes the expansion of the Universe, entropy production, affects the red-shift dependence of CMB, etc.

2) **The broken conformal symmetry may affect the rate of photons creation in the reheating period, leading to potentially observable consequences.**

3) **The conformal symmetry violation can be important for creation of initial seeds of magnetic field of the galaxies.**
Turner & Widrow, PRD (1988).
Dolgov, ZhETP (1981); PRD (1993).
Calzetta, Kandus & Mazzitelli, PRD (1998).
Bassett et al, PRD (2001).
Lambiase & Prasanna, PRD (2004).
Bamba & Sasaki, JCAP (2007).

4) **Quantum corrections can move the pole in the photon propagator and make the speed different from the one in the classical case.**

Similar effect occurs in curved space,

Khriplovich, PLB (1995) 251
Lafrance & Myers, PRD (1995).
Shore, NPB (1996,2001); Cont.Phys.(2003).
Dolgov & Novikov, PLB (1998).
Rong-Gen Cai, NPB (1998).
Mohanty & Prasanna, NPB (1998).
Scharnhorst, Ann.Phys. (1998).
Cougo-Pinto, Farina et al, PLB (1999).
Novello et al, PRD (2000).
Ferrer, de la Incera & Romeo, PLB (2001).
Bruneton, PRD (2007).
Balakin, Bochkarev & Lemos, PRD (2008).

The two main questions are as follows:

1) **What is the mechanism of violation of the local conformal symmetry at the quantum level?**

2) **To which extent the finite quantum corrections and, in particular, violation of local conformal symmetry, are universal?**

In other words, do we have an ambiguity in the quantum terms?

Consider the quantum effects of massive fields.

The one-loop effective action (EA) in the case of $g_{\mu\nu}$ and A_μ background can be defined via the path integral

$$e^{i\Gamma^{(1)}[g_{\mu\nu},\,A_\mu]} = \int D\psi D\bar{\psi}\, e^{iS_{QED}}\,,$$

or

$$\bar{\Gamma}^{(1)} = -\,\frac{1}{2}\,\mathrm{Log\ Det}\,\hat{H}\,,$$

where

$$\hat{H} = i\big(\gamma^\mu\nabla_\mu - iM - ie\gamma^\mu A_\mu\big)$$

We use heat-kernel method and the Schwinger-DeWitt technique in curved space QED.

Reducing the problem to the derivation of Log Det $\hat{\mathcal{O}}$,

$$\hat{\mathcal{O}} = \widehat{\Box} + 2\hat{h}^{\mu}\nabla_{\mu} + \hat{\Pi}\,.$$

Multiply $\hat{H}$ **by an appropriate conjugate** $\hat{H}^*$,

$$\hat{\mathcal{O}} = \hat{H} \cdot \hat{H}^*$$

and use the relation

$$\text{Log Det } \hat{H} = \text{Log Det } \hat{\mathcal{O}} - \text{Log Det } \hat{H}^*\,.$$

The simplest choice,

$$\hat{H}_1^* = -i\left(\gamma^\mu \nabla_\mu + iM - ie\gamma^\mu A_\mu\right).$$

According to G. De Berredo-Peixoto, M.Ph.L.A 16 (2001),

$$\text{Log Det } \hat{H} = \text{ Log Det } \hat{H}_1^* ,$$

then $$\text{Log Det } \hat{H} = \frac{1}{2} \text{ Log Det } \left(\hat{H}\hat{H}_1^*\right).$$

Our result for the one-loop Euclidean effective action is

$$\bar{\Gamma}^{(1)}_{\sim F^2} = \frac{e^2}{2(4\pi)^2} \int d^4x \sqrt{g}\, F_{\mu\nu} \left[\frac{2}{3\epsilon} + k_1^{FF}\right] F^{\mu\nu},$$

where $$k_1^{FF} = k_1^{FF}(a) = Y\left(2 - \frac{8}{3a^2}\right) - \frac{2}{9}.$$

$$Y = 1 - \frac{1}{a} \ln\left(\frac{2+a}{2-a}\right), \qquad a^2 = \frac{4\Box}{\Box - 4m^2}.$$

This expression represents a complete one-loop contribution.
Goncalves, de B.-Peixoto & I.Sh., hep-th/0904.4171

Is this result universal and unambiguous?

An aternative choice:

$$\hat{H}_2^* = -\,i\left(\gamma^\mu \nabla_\mu + iM\right).$$

This operator does not depend on A_μ, hence

$$\text{Log Det}\,\hat{H}\Big|_{FF} = \text{Log Det}\,\big(\hat{H}\hat{H}_2^*\big)\Big|_{FF}.$$

If the relation

$$\text{Det}\,(\hat{A}\cdot\hat{B}) = \text{Det}\,\hat{A}\cdot\,\text{Det}\,\hat{B}$$

holds in this case, we are going to meet the equal expressions,

$$\frac{1}{2}\,\text{Log Det}\,\big(\hat{H}\hat{H}_1^*\big)\Big|_{FF} = \text{Log Det}\,\big(\hat{H}\hat{H}_2^*\big)\Big|_{FF},$$

In reality, we meet

$$\frac{1}{2}\,\mathrm{Log\ Det}\,\big(\hat{H}\hat{H}_1^*\big)\Big|_{FF} \;=\; \mathrm{Log\ Det}\,\big(\hat{H}\hat{H}_2^*\big)\Big|_{FF}$$

for divergencies, but

$$\frac{1}{2}\,\mathrm{Log\ Det}\,\big(\hat{H}\hat{H}_1^*\big)\Big|_{FF} \;\neq\; \mathrm{Log\ Det}\,\big(\hat{H}\hat{H}_2^*\big)\Big|_{FF}$$

for nonlocal finite parts.

This is nothing else, but **the Multiplicative Anomaly (MA)**

Kontsevich & Vishik, hep-th/9406140/9404046.
Elizalde, Vanzo & Zerbini, Com.Math.Phys. 194 (1998);
Cognola, Elizalde and Zerbini, Com.Math.Phys. 237 (2003); NPB 532 (1998); ...

$$\bar{\Gamma}^{(1)}\Big|_{AA} = -\frac{e^2}{2(4\pi)^2}\int d^4x\sqrt{g}\,\Big\{F_{\mu\nu}\,\Big[\frac{2}{3\,\epsilon}+k_2^{FF}(a)\Big]F^{\mu\nu}$$

$$+\,2\nabla_\mu A^\mu\Big[Y\big(\frac{8}{3a^2}-2\big)+\frac{2}{9}\Big]\nabla_\nu A^\nu$$

$$+\,\nabla_\mu A^\nu\Big[\frac{16\,Y}{3a^2}+\frac{4}{9}\Big]\nabla_\nu A^\mu+\mathcal{O}(R\cdot A\cdot A)\Big\}\,,$$

where

$$k_2^{FF}(a) = Y\Big(1+\frac{4}{3a^2}\Big)+\frac{1}{9}\,,$$

and $\mathcal{O}(R\cdot A\cdot A)$ are terms proportional to scalar curvature.
Here ϵ is the parameter of dimensional regularization

$$\frac{1}{\epsilon} = \frac{2}{4-d}+\ln\Big(\frac{4\pi\mu^2}{m^2}\Big)-\gamma\,,\qquad \gamma = 0.5772\ldots\,.$$

Violation of gauge symmetry is an extra evidence of MA.

Why the MA takes place?

The heat kernel solution of our interest is a sum of the

$$\mathcal{O}(A^2) \,-\, \text{terms}$$

in all Schwinger-DeWitt coefficients $a_k = \text{Tr} \lim_{x' \to x} \hat{a}_k(x', x)$.

The general expressions for $\lim_{x' \to x} \hat{a}_k(x', x)$. **are universal,**

However their functional traces do depend on the dimensions n.

The divergences and therefore

$$a_1 \text{ in } n = 2\,, \quad a_2 \text{ in } n = 4\,,\,, \quad a_3 \text{ in } n = 6\,, \quad \text{etc}$$

are scheme independent. But the sum is of course scheme dependent for any particular n, including $n = 4$.

Conclusions.

1) The semiclassical approach to Quantum Gravity is interesting and worthwhile to study.

2) In the framework of this method we can learn a lot of relevant and interesting things about cosmology and related areas, such as black hole physics, and maybe even astronomy.

3) We have confirmed the existence of the Multiplicative Anomaly in QED, both in flat and curved space-time cases.

Different from other calclations, we are safe from the renormalization ambiguity problem.

End of the story