COMPLEX PROJECTION OF UNITARY DYNAMICS OF QUATERNIONIC PURE STATES

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1. Quantum mechanics on $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

In 1936, by using some lattice theoretic arguments, Birkhoff and von Neumann concluded that it is possible to consider the set of states of a quantum system as a vector space over the real, complex or quaternionic fields. While Stueckelberg showed that the real formulation of quantum mechanics is essentially equivalent to complex quantum mechanics (CQM), the research on quaternionic quantum mechanics (QQM) began much later with a series of papers by Finkelstein et al., in the sixties, and pursued up to now.

At present, the most serious problem of QQM concerns the description of compound systems, since in this theory the usual definition of Kronecker product of matrices does not hold, and also the standard definition of tensor product of Hilbert spaces cannot be used, owing to the non-commutativity of the skew-field $\mathbb{Q}$.

The possibility of a generalization of quantum mechanics based on quaternion field instead of complex field is still controversial. However, the rich structures emerging from such a generalization may be very useful in the description of entanglement, dynamical maps and decoherence phenomena in quantum physics.

The most general dynamics of the quantum state represented by a complex density matrix ρ_α can be described in terms of a dynamical map

$$\rho_\alpha \longrightarrow \mathcal{B}(\rho_\alpha). \tag{1}$$

The dynamical map represents the effect of the coupled (complex) unitary evolution of the system and its environment.

$$\rho_\alpha^A \longrightarrow \rho_\alpha^A(t) = \mathrm{Tr}_B[U\rho_\alpha^{AB}U^\dagger] = \mathcal{B}(\rho_\alpha^A), \tag{2}$$

where ρ_α^{AB} is the whole state of the system and environment and ρ_α^A is the state of the system obtained via partial trace.

If $\rho_\alpha^{AB} = \rho_\alpha^A \otimes \rho_\alpha^B$, then, $\mathcal{B}$ is completely positive (CP) i. e.,

$$\mathcal{B}\ \ CP \Leftrightarrow \mathcal{B} \otimes I_n \geq 0 \Leftrightarrow \mathcal{B}(\rho_\alpha^A) = \sum_s K_s \rho_\alpha^A K_s^\dagger. \tag{3}$$

Unitary maps U are CP.

The transposition $T : \rho_\alpha \longrightarrow \rho_\alpha^T$ is not CP.

Decomposable maps:

$$\mathcal{B} = \mathcal{B}_{CP}^1 + \mathcal{B}_{CP}^2 \circ T. \tag{4}$$

Any 2-dimensional positive map is decomposable.

A result by Kossakowski state that:

Any complex decomposable map can be obtained as the complex projection of completely positive quaternionic map on complex density matrices.

We apply such result to a very general framework of two qubits dynamics and we show that any positive complex dynamical map associated with such dynamics can be obtained as the complex projection of quaternionic unitary dynamics between quaternionic pure states.

2. Basic notation of QQM and density matrices

A quaternion is expressed as

$$q = q_0 + q_1 i + q_2 j + q_3 k \tag{5}$$

where $q_l \in \mathbb{R}$ $(l = 0, 1, 2, 3)$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$.

The quaternion skew-field $\mathbb{Q}$ is an algebra of rank 4 over $\mathbb{R}$, non commutative and endowed with an involutive anti-automorphism

$$q \rightarrow \bar{q} = q_0 - q_1 i - q_2 j - q_3 k. \tag{6}$$

In a (right) n-dimensional vector space $\mathbb{Q}^n$ over $\mathbb{Q}$, every linear operator is associated in a standard way with a $n \times n$ matrix acting on the left. Moreover, in analogy with the case of vector spaces over $\mathbb{C}$, one can introduce the concepts of unitarity, hermiticity and so on.

Every linear operator A can be written as $A = A_\alpha + jA_\beta$ where A_α and A_β are complex matrices.

In QQM, the Schrödinger equation becomes

$$\frac{d}{dt}|\psi\rangle = -H|\psi\rangle, \tag{7}$$

where H is the anti-hermitian quaternionic Hamiltonian operator.

The density matrix ρ_ψ associated with a pure state $|\psi\rangle$ belonging to $\mathbb{Q}^n$ is defined by

$$\rho_\psi = |\psi\rangle\langle\psi|. \tag{8}$$

Quaternionic mixed states are described by positive quaternionic hermitian operators (density matrices) ρ on $\mathbb{Q}^n$ with unit trace and rank greater than one.

The expectation value of a quaternionic hermitian operator A on a state $|\psi\rangle$ can be expressed in terms of ρ_ψ as

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle = Re\mathrm{Tr}(A|\psi\rangle\langle\psi|) = Re\mathrm{Tr}(A\rho_\psi). \tag{9}$$

Expanding $A = A_\alpha + jA_\beta$ and $\rho = \rho_\alpha + j\rho_\beta$ in terms of complex matrices A_α, A_β, ρ_α and ρ_β, it follows that the expectation value $\langle A\rangle_\psi$ may depend on A_β or ρ_β only if both A_β and ρ_β are different from zero. Indeed,

$$\langle A\rangle_\rho = Re\mathrm{Tr}(A\rho) = Re\mathrm{Tr}(A_\alpha\rho_\alpha - A^*_\beta\rho_\beta), \tag{10}$$

where $*$ denotes complex conjugation.

Thus, the expectation value of an hermitian operator A on the state ρ depends on the quaternionic parts of A and ρ, only if both the observable and the state are represented by genuine quaternionic matrices.

However, if an observable O is described by a pure *complex* hermitian matrix, its expectation value does not depend on the quaternionic part $j\rho_\beta$ of the state $\rho = \rho_\alpha + j\rho_\beta$.

Moreover, *the expectation value predicted in the standard (complex) Quantum Mechanics for the state ρ_α coincides with the one predicted in Quaternionic Quantum Mechanics for the state ρ,* since $\mathrm{Tr}(\mathrm{O}\rho_\alpha) = Re\mathrm{Tr}(O\rho_\alpha) = Re\mathrm{Tr}(O\rho)$.

This observation enables us to merge CQM in the framework of QQM, without modifying any theoretical prediction, as long as complex observables are taken into account.

Let us denote by $M(\mathbb{Q})$ and $M(\mathbb{C})$ the space of $n \times m$ quaternionic and complex matrices respectively and let $M = M_\alpha + jM_\beta \in M(\mathbb{Q})$. We define the complex projection

$$P : M(\mathbb{Q}) \to M(\mathbb{C}) \tag{11}$$

by the relation

$$P[M] = \frac{1}{2}[M - iMi] = M_\alpha. \tag{12}$$

The probability P_c^ρ that a quaternionic state $\rho = \rho_\alpha + j\rho_\beta$ is complex can be defined as follows:

$$P_c^\rho := Re\mathrm{Tr}\left(P[\rho]\rho\right) = \mathrm{Tr}\left(\rho_\alpha^2\right). \tag{13}$$

When we consider time-dependent quaternionic unitary dynamics,

$$\rho(t) = U(t)\rho(0)U^{\dagger}(t), \tag{14}$$

where

$$U(t) = (U_\alpha + jU_\beta)(t) = T_o\, e^{-\int_0^t du H(u)} \tag{15}$$

and T_o denotes the time ordering operator, the differential equation associated with the time evolution for $\rho = \rho_\alpha + j\rho_\beta$ reads

$$\frac{d}{dt}\rho(t) = -[H(t), \rho(t)] \tag{16}$$

where $H(t) = H_\alpha + jH_\beta = -\left(\frac{d}{dt}U(t)\right)U^{\dagger}(t)$.

The complex projection of these equations reads

$$\rho_\alpha(t) = U_\alpha\rho_\alpha(0)U_\alpha^{\dagger} + U_\beta^*\rho_\alpha^*(0)U_\beta^T + U_\alpha\rho_\beta^*(0)U_\beta^T - U_\beta^*\rho_\beta(0)U_\alpha^{\dagger} \tag{17}$$

and

$$\frac{d}{dt}\rho_\alpha = -[H_\alpha, \rho_\alpha] + H_\beta^*\rho_\beta - \rho_\beta^* H_\beta. \tag{18}$$

3. The complex projection of quaternionic density matrices

We focus now on the complex projection ρ_α of quaternionic density matrices $\rho = \rho_\alpha + j\rho_\beta$.

From the hermiticity of ρ and ρ_α we get

$$\mathrm{Tr}\rho_\alpha = Re\mathrm{Tr}\rho_\alpha = Re\mathrm{Tr}\rho = \mathrm{Tr}\rho, \tag{19}$$

i. e., the complex projection of quaternionic density matrices is trace preserving. Moreover, the following proposition holds

Proposition 0. *the complex projection ρ_α of any quaternionic density matrix $\rho = \rho_\alpha + j\rho_\beta$ is a complex density matrix.*

The following statement give information about the rank of the complex projection ρ_α:

Proposition 1. *Let $\rho = \rho_\alpha + j\rho_\beta$ be a n-dimensional quaternionic density matrix, and let $\mathrm{rank}\,\rho = m$. Then, $m \leq \mathrm{rank}\,\rho_\alpha \leq 2m$.*

The converse result also holds:

Proposition 2. *Let ρ_α be a n-dimensional complex density matrix with $\mathrm{rank}\,\rho_\alpha = m > 1$ and let $[x]$ denote the integer part of x.Then, for any m' with $\left[\frac{m+1}{2}\right] \leq m' \leq m$ there exists a (skew-symmetric) complex matrix ρ_β such that $\rho = \rho_\alpha + j\rho_\beta$ is a density matrix with $\mathrm{rank}\,\rho = m'$.*

As a consequence of the above propositions, we can conclude that:

Any complex density matrix ρ_α can be obtained as the complex projection of a quaternionic pure density matrix $\rho = \rho_\alpha + j\rho_\beta$ if and only if $\mathrm{rank}\rho_\alpha = 2$.

Now, let us consider an arbitrary pair of complex density matrices, ρ_α and ρ'_α such that $\operatorname{rank}\rho_\alpha \leq 2$ and $\operatorname{rank}\rho'_\alpha \leq 2$, and let be $\mathcal{B}$ a complex dynamical map,

$$\mathcal{B} : \rho_\alpha \rightarrow \rho'_\alpha = \mathcal{B}(\rho_\alpha). \tag{20}$$

According with proposition 2 we can "purify" the complex states ρ_α and ρ'_α by adding suitable purely quaternionic terms $j\rho_\beta$ and $j\rho'_\beta$ respectively. Moreover, since any pair of quaternionic hermitian matrices admitting the same eigenvalues are unitary equivalent, we immediately obtain that the map $\mathcal{B}$ can be described as the complex projection of a quaternionic unitary map between quaternionic pure states $\rho = \rho_\alpha + j\rho_\beta$ and $\rho' = \rho'_\alpha + j\rho'_\beta$:

$$U : \rho \rightarrow \rho' = U\rho U^\dagger, \quad UU^\dagger = U^\dagger U = \mathbf{1}, \tag{21}$$

where $\rho'_\alpha = \mathcal{B}(\rho_\alpha) = P[\rho']$.

In this way, dynamical maps of (complex) quantum mechanical systems can be interpreted in terms of the complex projection of unitary dynamics between quaternionic pure states whenever the rank of their complex density matrices is less or equal than two.

Our approach can also be applied to higher dimensional bipartite and multipartite quantum systems whenever the rank of the complex density matrices of their components, obtained via partial traces, is not higher than two for any time.

4. Mixed $\mathbb{C}$-qubits and pure $\mathbb{Q}$-qubits

Because of the relevance of two qubit quantum gates in quantum information processing, we shall consider now the dynamical maps for the reduced unitary evolution of two $\mathbb{C}$-qubits and we describe them as the complex projections of unitary dynamics between pure $\mathbb{Q}$-qubits.

According with propositions 1 and 2, any complex (mixed) state

$$\rho_\alpha = \frac{1}{2}\begin{pmatrix} 1+a_3 & a_1 - ia_2 \\ a_1 + ia_2 & 1 - a_3 \end{pmatrix}, \quad (a_i \in \mathbb{R}, \quad 1 - a_1^2 - a_2^2 - a_3^2 > 0) \quad (22)$$

can be purified, by adding the purely quaternionic hermitian term

$$j\rho_\beta = j\frac{e^{-i\theta}\sqrt{1-a_1^2-a_2^2-a_3^2}}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta \in \mathbb{R}. \quad (23)$$

The system we will study is composed of two qubits A and B, parametrized by the Bloch vectors $\overrightarrow{a} \equiv (a_1, a_2, a_3)$ and $\overrightarrow{b} \equiv (b_1, b_2, b_3)$. The most general Hamiltonian for two qubits can be written as

$$H = \sum_i \gamma_i \sigma_i^A \otimes \sigma_i^B \quad (24)$$

where γ_1, γ_2 and γ_3 are constant. The evolution operator U of the overall state ρ^{AB} assumes the simple form:

$$U = \prod_{j=1}^{3}[\cos(\gamma_j t)\mathbb{I}^A \otimes \mathbb{I}^B - i\sin(\gamma_j t)\sigma_j^A \otimes \sigma_j^B]. \quad (25)$$

4.1. Optimal Entanglement generation

An interesting regime to study is related to the creation of maximally entangled Bell states. Assume that we have initially two complex pure states, with Bloch vectors $\overrightarrow{a} \equiv (1,0,0)$ and $\overrightarrow{b} \equiv (0,1,0)$ respectively. Assume

$$\gamma_3 = 1, \quad \gamma_1 = \gamma_2 = 0. \tag{26}$$

The initial complex $\mathbb{C}$-qubits are given by

$$\rho_\alpha^A(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \rho_\alpha^B(0) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \tag{27}$$

and the evolution operator of the overall system assumes the form

$$U = \cos t \mathbf{1}^A \otimes \mathbf{1}^B - i \sin t \sigma_3^A \otimes \sigma_3^B = \tag{28}$$

$$\begin{pmatrix} e^{-it} & 0 & 0 & 0 \\ 0 & e^{it} & 0 & 0 \\ 0 & 0 & e^{it} & 0 \\ 0 & 0 & 0 & e^{-it} \end{pmatrix}. \tag{29}$$

Then, we obtain by partial traces the final states

$$\begin{aligned} \rho_\alpha^A(t) &= \frac{1}{2}\begin{pmatrix} 1 & \cos 2t \\ \cos 2t & 1 \end{pmatrix}, \\ \rho_\alpha^B(t) &= \frac{1}{2}\begin{pmatrix} 1 & -i\cos 2t \\ i\cos 2t & 1 \end{pmatrix}. \end{aligned}$$

At the time $t_{bell} = \pi/4$, the purity of each $\mathbb{C}$-qubit goes to a minimum and the overall state $\rho^{AB}(t = t_{bell})$ is equivalent to a Bell state

$$\frac{1}{\sqrt{2}}(|00> + |11>). \tag{30}$$

Let us describe this dynamics as the complex projection of a quaternionic unitary dynamics between quaternionic pure $\mathbb{Q}$-qubits. By purification the initial and final pure $\mathbb{Q}$-qubits read:

$$\rho^A(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \rho^B(0) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \tag{31}$$

and

$$\rho^A(t) = \frac{1}{2}\begin{pmatrix} 1 & \cos 2t \\ \cos 2t & 1 \end{pmatrix} + j\frac{e^{-i\varphi}}{2}\begin{pmatrix} 0 & -\sin 2t \\ \sin 2t & 0 \end{pmatrix}, \ \varphi \in \mathbb{R} \tag{32}$$

$$\rho^B(t) = \frac{1}{2}\begin{pmatrix} 1 & -i\cos 2t \\ i\cos 2t & 1 \end{pmatrix} + j\frac{e^{-i\theta}}{2}\begin{pmatrix} 0 & -\sin 2t \\ \sin 2t & 0 \end{pmatrix}, \ \theta \in \mathbb{R}. \tag{33}$$

The quaternionic unitary evolution operators reads

$$U^A(t) = \begin{pmatrix} \cos 2t + je^{-i\varphi}\sin 2t & 0 \\ 0 & 1 \end{pmatrix}, \tag{34}$$

$$U^B(t) = \begin{pmatrix} \cos 2t + ke^{-i\theta}\sin 2t & 0 \\ 0 & 1 \end{pmatrix}. \tag{35}$$

The corresponding anti-hermitian quaternionic Hamiltonians turn out o be constant:

$$H^A(t) = -\left(\frac{d}{dt}U^A(t)\right)U^{A\dagger}(t) = \begin{pmatrix} -2je^{-i\varphi} & 0 \\ 0 & 0 \end{pmatrix} \tag{36}$$

and

$$H^B(t) = -\left(\frac{d}{dt}U^B(t)\right)U^{B\dagger}(t) = \begin{pmatrix} -2ke^{-i\theta} & 0 \\ 0 & 0 \end{pmatrix}. \tag{37}$$

The expectation value of the energy observables $|H^{A,B}(t)|$ on the states $\rho^{A,B}(t)$ reads

$$\langle |H^{A,B}(t)| \rangle_{\rho^{A,B}(t)} = Re\mathrm{Tr}\left(|H^{A,B}(t)|\rho^{A,B}(t)\right) = 1. \tag{38}$$

We remark that the quaternionic unitary operators $U^A(t)$ and $U^B(t)$ satisfy a one-parameter semigroup composition law: $U(t)U(t') = U(t+t')$ for all t, t'. The time evolution of the density matrices $\rho_\alpha^{A,B}(t)$ is ruled by the differential equations:

$$\frac{d}{dt}\rho_\alpha^A(t) = -[H_\alpha^A, \rho_\alpha^A] + H_\beta^{A*}\rho_\beta^A - \rho_\beta^{A*}H_\beta^A = \sin 2t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{39}$$

$$\frac{d}{dt}\rho_\alpha^B(t) = -[H_\alpha^B, \rho_\alpha^B] + H_\beta^{B*}\rho_\beta^B - \rho_\beta^{B*}H_\beta^B = \sin 2t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{40}$$

The probabilities $P_c^{\rho^A}(t)$ and $P_c^{\rho^B}(t)$ that the quaternionic states $\rho^A(t)$ and $\rho^B(t)$ are complex are respectively given by

$$P_c^{\rho^A}(t) = Re\mathrm{Tr}\left(P[\rho^A(t)]\rho^A(t)\right) = \frac{1}{2}(1 + (\cos 2t)^2) \tag{41}$$

$$P_c^{\rho^B}(t) = Re\mathrm{Tr}\left(P[\rho^B(t)]\rho^B(t)\right) = \frac{1}{2}(1 + (\cos 2t)^2) \tag{42}$$

and coincide with the probabilities that the reduced complex density matrices $\rho_\alpha^A(t)$ and $\rho_\alpha^B(t)$ become pure, since at the same time the quaternionic terms $\rho_\beta^{A,B}$ vanish.

5. Concluding remarks

The main implication of the above results is very surprising.

For any compound system made of N $\mathbb{C}$-qubits, each subsystem can be described by a pure $\mathbb{Q}$-qubit, which undergoes a unitary quaternionic time evolution.

Hence, one can attribute to each subsystem "individual" properties, differently from what happens in the realm of CQM where reduced density matrices do not allow a similar interpretation.

Nevertheless, the correlations between subsystems do not disappear at all, but they are implicitly taken into account in such individual evolutions, as the example analyzed above point out.

This results point out to an apparently puzzling situation, in which the same state of a physical system is entangled in CQM, while it seems to be "separable" in QQM.

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