Quantum fluctuations of exotic Kinks

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Outline

1. Deformed $\mathbb{O}(N)$ linear sigma models
   - Vacuum fluctuations
   - Kink fluctuations
   - One-loop kink mass shift
   - N=1: the $\lambda(\phi^4)_2$ model
   - N=2: the Bazeia-Nascimento-Ribeiro-Toledo model
   - N=3: the massive non-linear $\mathbb{S}^2$-sigma model

The General Model

ACTION FUNCTIONAL: $S[\phi_1, \phi_2, \cdots, \phi_N]$,

$$S = \frac{m^2}{\lambda} \int dx^2 \left\{ \frac{1}{2} \sum_{a=1}^{N} \frac{\partial \phi_a}{\partial x^\mu} \cdot \frac{\partial \phi_a}{\partial x_\mu} - V(\phi_a; \sigma_{ab}, \sigma_a^2) \right\}$$

POTENTIAL ENERGY DENSITY:

$V(\phi_1(x^\mu), \phi_2(x^\mu), \cdots, \phi_N(x^\mu))$,

$$V = \frac{1}{2} \left(\sum_{a=1}^{N} \phi_a \phi_a - 1 \right)^2 + \sum_a \sum_b{}_{\, a \leq b} \sigma_{ab} \phi_a^2 \phi_b^2 + \sum_{a=1}^{N} \sigma_a^2 \phi_a \phi_a$$

Outline

Classical minima

THERE EXIST 2^N CLASSICAL MINIMA:

- Parameter range:

$$(a) \qquad 1 > \sigma_a^2 > -\infty$$

$$(b) \qquad \sigma_{ab} > \max\left(\frac{1-\sigma_a^2}{1-\sigma_b^2}(1+2\sigma_{bb}), \frac{1-\sigma_b^2}{1-\sigma_a^2}(1+2\sigma_{aa})\right) > 0\,,\ a \neq b$$

$$(c) \qquad 1+2\sigma_{aa} > 0$$

- Derivative of the potential:

$$\frac{\partial V}{\partial \phi_a} = 2\phi_a \left[\sum_{b=1}^{N} \phi_b\phi_b + \sum_{a \leq b} \sigma_{ab}\phi_b^2 + \sigma_a^2 - 1\right]$$

- Minima of the potential:

$$\frac{\partial V}{\partial \phi_a}\Big|_{V^{(a)}} = 0 \quad \Rightarrow \quad \phi_b^{V^{(a)}} = 0\,,\ b \neq a \quad , \quad \phi_a^{V^{(a)}} = \pm\sqrt{\frac{1-\sigma_a^2}{1+2\sigma_{aa}}}$$

Vacuum States

- TRUE VACUA $\equiv$ Absolute minima:

$$V(\phi_c^{V^{(c)}}) = \frac{\sigma_c^2(2-\sigma_c^2)+2\sigma_{cc}}{2(1+2\sigma_{cc})}$$

- FALSE VACUA $\equiv$ Relative minima: $V(\phi_a^{V^{(a)}}) > V(\phi_c^{V^{(c)}})$

$$V^{(a)} \equiv (0,0,\cdots,\pm\sqrt{\frac{1-\sigma_a^2}{1+2\sigma_{aa}}},\cdots,0,0)$$

Small vacuum fluctuations

- Small fluctuations: $\phi_a(t,x) = \phi_a^{V^{(c)}} + \eta_a(t,x) \quad ; \quad x_0 = t\,,\, x_1 = x$
- Action $S^{(2)}[\phi_a^{V^{(c)}};\eta_1,\cdots,\eta_N]$:

$$S^{(2)} = \frac{m^2}{2\lambda}\int dx^2 \sum_{a=1}^{N}\left[\frac{\partial\eta_a}{\partial t}\frac{\partial\eta_a}{\partial t} - \eta_a \underbrace{\left(-\frac{\partial^2}{\partial x^2} + \mu_a^2\right)}_{K_0}\eta_a\right] + \mathcal{O}(\eta^3)$$

- Hessian operator:

$$K_0 = \begin{pmatrix} -\frac{d^2}{dx^2}+\mu_1^2 & \cdots & \cdots & 0 \\ 0 & -\frac{d^2}{dx^2}+\mu_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & -\frac{d^2}{dx^2}+\mu_N^2 \end{pmatrix}$$

with

$$\mu_a^2 = \left.\frac{\partial^2 V}{\partial\phi_a^2}\right|_{V^{(c)}} = 2\left[\frac{1-\sigma_c^2}{1+2\sigma_{cc}}(1+\sigma_{ac}) - (1-\sigma_a^2)\right] \;,\; a \neq c$$

$$\mu_c^2 = \left.\frac{\partial^2 V}{\partial\phi_c^2}\right|_{V^{(c)}} = 4(1-\sigma_c^2)$$

Vacuum states

- Sum over modes with PBC: $f^a(x+l) = f^a(x) \quad , \; l = \frac{mL}{\sqrt{2}}$

$$K_0 \text{ acts on} : L^2 = \bigoplus_{a=1}^N L^2_a(\mathbb{S}^1)$$

- Eigenvalues problem:

$$K_0 f_n^a(x) = \omega_a^2(k_n) f_n^a(x) \quad \Rightarrow \quad \begin{cases} \omega_a^2(k_n) = k_n^2 + \mu_a^2 \\ k_n = \dfrac{2\pi}{l} n \quad , \quad n \in \mathbb{Z} \\ f_n^a(x) = \dfrac{1}{\sqrt{l}} e^{ik_n x} \end{cases}$$

- Free Hamiltonian $[\hat{b}_a^\dagger(k_n), \hat{b}_c(k_m)] = \delta_{ac}\delta_{mn}$

$$\hat{H}^{(2)} = \frac{\hbar m}{\sqrt{2}} \sum_{a=1}^N \sum_{n\in\mathbb{Z}} \omega_a(k_n) \left(\hat{b}_a^\dagger(k_n) \hat{b}_a(k_n) + \frac{1}{2} \right)$$

- Vacua as coherent states:

$$b_a(k_n)|0;V\rangle = 0 \; , \forall a, \forall k_n \quad \Rightarrow \quad \begin{cases} \hat{\phi}_a(t,x)|0;V\rangle = 0 \, , \; a \neq c \\ \hat{\phi}_c(t,x)|0;V\rangle = \phi_c^{V^{(c)}} \hat{\phi}_a(t,x)|0;V\rangle \end{cases}$$

Vacuum states

- Vacuum energy

$$\langle 0;V|\hat{H}^{(2)}|0;V\rangle = \frac{\hbar m}{2\sqrt{2}}\mathrm{Tr}_{L^2}K_0^{\frac{1}{2}}$$

- SPECTRAL ZETA FUNCTION REGULARIZATION: set the value of the spectral zeta function at a regular point $s\in\mathbb{C}$.

$$\zeta_{K_0}(s)=\mathrm{Tr}_{L^2}K_0^{-s}=\sum_{a=1}^{N}\sum_{n=-\infty}^{\infty}\frac{1}{(\frac{4\pi^2}{l^2}n^2+\mu_a^2)^s}=\sum_{a=1}^{N}E(s,\mu_a^2|\frac{4\pi^2}{l^2})$$

is a sum of Epstein zeta functions (Meromorphic functions of s).

- K_0-HEAT TRACE:

$$\mathrm{Tr}_{L^2}e^{-\beta K_0}=\sum_{a=1}^{N}\sum_{n\in\mathbb{Z}}e^{-\beta(\frac{4\pi^2}{l^2}n^2+\mu_a^2)}=\underline{\Theta\begin{bmatrix}0\\0\end{bmatrix}(0|i\frac{4\pi}{l^2}\beta)\cdot\sum_{a=1}^{N}e^{-\beta\mu_a^2}}$$

$$\Theta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau)=\sum_{n\in\mathbb{Z}}e^{2\pi i(n+a)(z+b)+\frac{(n+a)^2}{2}\tau)}$$

$$a,b=0,\tfrac{1}{2}\quad,\quad z\in\mathbb{C}\quad,\quad\tau\in\mathbb{C}\,,\ \mathrm{Im}\tau>0$$

Vacuum states

- **MODULAR TRANSFORMATION**

$$\tau = i\frac{4\pi}{l^2}\beta \to -\frac{1}{\tau} = i\frac{l^2}{4\pi\beta}$$

$$\mathrm{Tr}_{L^2} e^{-\beta K_0} = \frac{l}{\sqrt{4\pi\beta}} \cdot \Theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](0|i\frac{l^2}{4\pi\beta}) \cdot \sum_{a=1}^{N} e^{-\beta\mu_a^2}$$

$$\Theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](0|i\frac{l^2}{4\pi\beta}) = \sum_{n\in\mathbb{Z}} e^{-\frac{l^2}{4\beta}n^2} \cong_{\beta\to 0} 1 + \mathcal{O}(e^{-\frac{c}{\beta}})$$

- **MELLIN TRANSFORM**

$$\zeta_{K_0}(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\beta\, \beta^{s-1}\, \mathrm{Tr}_{L^2} e^{-\beta K_0}$$

$$\begin{aligned}\zeta_{K_0}(s) &= \frac{1}{\Gamma(s)} \cdot \int_0^\infty d\beta\, \beta^{s-1} \sum_{a=1}^{N} e^{-\mu_a^2\beta} \left(\frac{l}{\sqrt{4\pi}}\beta^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\frac{l^2}{\beta}n^2}\right) \\ &= \frac{l}{\sqrt{4\pi}\Gamma(s)} \cdot \sum_{a=1}^{N} \left(\frac{\Gamma(s-\frac{1}{2})}{\mu_a^{2s-1}} + 2\sum_{n\in\mathbb{Z}/\{0\}} \left(\frac{nl}{\mu_a}\right)^{s-\frac{1}{2}} K_{\frac{1}{2}-s}(2\mu_a nl)\right)\end{aligned}$$

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Kink ground states

• TK1 (one-component topological kinks) kinks

Choose $V^{(c)}$:

$$V(\phi_c^{V^{(c)}}) = \frac{\sigma_c^2(2-\sigma_c^2)+2\sigma_{cc}}{2(1+2\sigma_{cc})}$$

Trial orbit: $\phi_a^{K^{(c)}} = 0 \quad , \quad \forall a \neq c$

First-order equations:

$$\frac{d\phi_c}{dx} = \pm\sqrt{2\left(V(\phi_c) - V(\phi_c^{V^{(c)}})\right)}$$

Kink solutions:

$$\phi_c^{K^{(c)}}(x) = \pm\phi_c^{V^{(c)}}\tanh[\frac{\mu_c}{2}(x-x_0)]$$

Finite kink energy

$$2\frac{m^3}{\sqrt{2}\lambda}\int_{-\infty}^{\infty} dx\left[V(\phi_c^{K^{(c)}}) - V(\phi_c^{V^{(c)}})\right] = \frac{4}{3}\frac{1}{\sqrt{1-\sigma_c^2}}\frac{m^3}{\sqrt{2}\lambda}$$

Kink ground states

- Kink small fluctuations, quadratic approximation

$$\begin{aligned} \phi_a(t,x) &= \phi_a^{K^{(c)}}(x) + \eta_a(t,x) \\ S^{(2)}[\phi_a^{K^{(c)}}(x);\eta_1,\cdots,\eta_N] &= \frac{m^2}{2\lambda}\int dx^2 \sum_{a=1}^{N}\left[\frac{\partial\eta_a}{\partial t}\frac{\partial\eta_a}{\partial t} - \eta_a K \eta_a\right] + \mathcal{O}(\eta^3) \end{aligned}$$

- Kink Hessian operator

$$\begin{aligned} K &= \mathrm{diag}\,(K_{11}, K_{22}, \cdots, K_{cc}, \cdots, K_{NN}) \\ K_{cc} &= -\frac{d^2}{dx^2} + \mu_c^2 - \frac{3}{2}\mu_c^2 \cdot \mathrm{sech}^2[\frac{\mu_c}{2}x] \\ K_{aa} &= -\frac{d^2}{dx^2} + \mu_a^2 - 2(1+\sigma_{ac})\left(\phi_c^{V^{(c)}}\right)^2 \cdot \mathrm{sech}^2(\frac{\mu_c}{2}x) \end{aligned}$$

Kink ground states

- Sum over modes with PBC: again $f^a(x+l) = f^a(x)$

$$K \text{ acts on} : L^2 = \bigoplus_{a=1}^{N} L^2_a(\mathbb{S}^1)$$

$$K_{cc} f^c_n(x) = \varepsilon^2_c(k_n) f^c_n(x)$$

Eigenvalues	Eigenfunctions
$\varepsilon^2_c(0) = 0$	$f^c_0(x) = \text{sech}^2[\frac{\mu_c}{2}x]$
$\varepsilon^2_c(3) = \frac{3}{4}\mu^2_c$	$f^c_3(x) = \sinh[\frac{\mu_c}{2}x]\text{sech}^2[\frac{\mu_c}{2}x]$
$\varepsilon^2_c(k^2) = k^2 + \mu^2_c$	$f^c_k(x) = e^{ikx}(3\tanh^2[\frac{\mu_c}{2}x] - 1 - \frac{6}{\mu_c} ik\tanh[\frac{\mu_c}{2}x] - \frac{4}{\mu^2_c}k^2)$

Phase shifts

$$f^c_k(x) \cong_{x\to\infty} = e^{i(kx + \frac{1}{2}\delta(k))} \qquad \Rightarrow \qquad \delta(k) = -2\arctan\frac{3\mu_c k}{\mu^2_c - 2k^2}$$

Periodic boundary conditions:

$$k_n l + \delta(k_n) = 2\pi n \;, \qquad , \qquad k_n - \frac{2\pi}{l} n = \frac{2}{l}\cdot\arctan\frac{3\mu_c k_n}{\mu^2_c - 2k^2_n} \qquad , \qquad n \in \mathbb{Z}$$

Spectral density

$$\rho_K(k) = \frac{1}{2\pi}\left(l + \frac{d\delta(k)}{dk}\right)$$

Kink ground states

$$K_{aa}f_n^a(x)=\varepsilon_a^2(k_n)f_n^a(x),\ a\neq c$$

$A=\sqrt{\frac{1+\sigma_{ac}}{1+2\sigma_{cc}}+\frac{1}{4}}$. $j=0,1,2,\cdots;I[A-\frac{1}{2}]$: number of bound states

Eigenvalues	Eigenfunctions
$\varepsilon_a^2(j)=\left(\mu_a^2-\frac{\mu_c^2}{4}\left(A-(j+\frac{1}{2})\right)^2\right)$	$f_j^a(x)=[\text{sech}(\frac{\mu_c}{2}x)]^{j+\frac{1}{2}-A}{}_2F_1[-j+2a,-j,\frac{1}{2}-j+A;z]$
$\varepsilon_a^2(k^2)=k^2+\mu_a^2$	$f_k^a(x)=[\text{sech}(\frac{\mu_c}{2}x)]^{ik}{}_2F_1[\frac{1}{2}-ik+A,\frac{1}{2}-ik+A,1-ik;z]$

$z=\frac{1}{2}(1+\tanh(\frac{\mu_c}{2}x))$

Reflection and transmission coefficients

$$T_a(k)=\frac{\Gamma(\frac{1}{2}-ik+A)\Gamma(\frac{1}{2}-ik-A)}{\Gamma(1-ik)\Gamma(-ik)}\quad,\quad R_a(k)=\frac{T_a(k)\Gamma(1-ik)\Gamma(ik)}{\Gamma(\frac{1}{2}-A)\Gamma(\frac{1}{2}+A)}$$

Phase shifts

$$\delta_a(k)=\delta_a^+(k)+\delta_a^-(k)\quad;\quad\delta_a^\pm(k)=\frac{1}{4}\arctan\left(\frac{\text{Im}(T_a(k)\pm R_a(k))}{\text{Re}(T_a(k)\pm R_a(k))}\right)$$

Kink ground states

- Spectral densities

$$\rho_{K_{aa}}(k) = \frac{1}{2\pi}(l + \frac{d\delta_a}{dk}(k))$$

Temporal **ASSUMPTION**: $\varepsilon_a(j) > 0, \forall a$ $\equiv$ TK1 kink **ISOLATED** and **STABLE**

- Free Hamiltonian

$$\hat{H}^{(2)} = \frac{\hbar m}{\sqrt{2}} \sum_{a=1}^{N} \left[\sum_{j=1}^{N_b(a)} \varepsilon_a(j) \left(\hat{B}_a^\dagger(j)\hat{B}_a(j) + \frac{1}{2} \right) + \sum_{n\in\mathbb{Z}} \varepsilon_a(k_n) \left(\hat{B}_a^\dagger(k_n)\hat{B}_a(k_n) + \frac{1}{2} \right) \right]$$

$$[\hat{B}_a^\dagger(k_n), \hat{B}_c(k_m)] = \delta_{ac}\delta_{mn} \quad , \quad [\hat{B}_a^\dagger(j), \hat{B}_c(l)] = \delta_{ac}\delta_{jl} \quad , \quad [\hat{B}_a^\dagger(j), \hat{B}_b(k_n)] = 0$$

Kink ground states

- Kink coherent states

$$\hat{B}_a(k_n)|0;K\rangle = 0\,, \quad \forall a\,, \, \forall k_n \qquad \Rightarrow \qquad \begin{cases} \hat{\phi}_a(t,x)|0;K\rangle = 0\,, a\neq c \\ \hat{\phi}_c(t,x)|0;K\rangle = \tanh[\frac{\mu_c}{2}x]|0;K\rangle \end{cases}$$

- Kink ground state energy

$$\langle 0;K|\hat{H}^{(2)}|K;0\rangle = \frac{\hbar m}{2\sqrt{2}}\mathrm{Tr}_{L^2}K^{\frac{1}{2}}$$

- Kink spectral zeta function for very very large l

$$\zeta_K(s) = \mathrm{Tr}_{L^2}K^{-s} = \sum_{a=1}^{N}\left(\sum_{j=0}^{N_b(a)}\frac{1}{\varepsilon_a^s(j)} + \int_{-\infty}^{\infty} dk\, \rho_{K_{aa}}(k)\frac{1}{(k^2+\mu_a^2)^{\frac{s}{2}}}\right)$$

REMARK: zero modes, $\varepsilon_c(0) = 0$, do not enter

Outline

Kink Casimir energy

• Zero point renormalization

$$\begin{aligned}\triangle E_K^C(s) &= \left(\triangle E_K(s) - \triangle E_{K_0}(s)\right) \\ &= 2^{s-1}\hbar\left(\frac{\mu^2}{m^2}\right)^s \mu\left(\zeta_K(s) - \zeta_{K_0}(s)\right)\end{aligned}$$

$$\triangle E_K^C = \lim_{s\to -\frac{1}{2}} \triangle E_K^C(s) = \frac{\hbar m}{2\sqrt{2}}\left(\zeta_K(-\frac{1}{2}) - \zeta_{K_0}(-\frac{1}{2})\right)$$

• Ultraviolet divergence of the two-field contraction at the same point (one-loop): $\hat{\phi}_a^2(x^\mu) =: \hat{\phi}_a^2(x^\mu) : +\delta\mu_a^2$

$$\delta\mu_a^2 = \int \frac{dk}{4\pi}\frac{1}{\sqrt{k^2+\mu_a^2}} = \frac{1}{2l}\sum_{n\in\mathbb{Z}}\frac{1}{(\frac{4\pi^2}{l^2}n^2+\mu_a^2)^{1/2}}$$

Mass renormalization Kink energy

- Normal ordering through Wick's theorem

$$\hat{\mathcal{H}}(\hat{\phi}_1(x^\mu),\cdots,\hat{\phi}_N(x^\mu)) = \sum_{a=1}^N \frac{1}{2}\left(\frac{\partial\hat{\phi}_a}{\partial t}\frac{\partial\hat{\phi}_a}{\partial t} + \frac{\partial\hat{\phi}_a}{\partial x}\frac{\partial\hat{\phi}_a}{\partial x}\right) + $$
$$+\hat{V}(\hat{\phi}_1(x^\mu),\cdots,\hat{\phi}_N(x^\mu))$$

$$:\hat{\mathcal{H}} := \hat{\mathcal{H}} + :\left(1-\exp[-\hbar\widehat{\sum_{a=1}^N \delta\mu_a^2\frac{\delta^2}{\delta\phi_a^2}}]V\right) := \hat{\mathcal{H}} + \hbar\sum_{a=1}^N \delta\mu_a^2 : \widehat{\frac{\delta^2 V}{\delta\phi_a^2}} : +\mathcal{O}(\hbar^2)$$

- Zeta function regularization

$$\delta\mu_a^2(s) = -\frac{1}{l}\frac{\Gamma(s+1)}{\Gamma(s)}\sum_{a=1}^N \zeta_{K_{0aa}}(s+1)$$

$$\begin{aligned}\triangle E_K^R(s) &= \hbar\sum_{a=1}^N \delta\mu_a^2\int dx\left(\langle 0;K| : \widehat{\frac{\delta^2 V}{\delta\phi_a^2}} : |K;0\rangle - \langle 0;V| : \widehat{\frac{\delta^2 V}{\delta\phi_a^2}} : |V;0\rangle\right)\\ &= -\lim_{l\to\infty}\frac{\hbar}{2l}\left(\frac{2\mu^2}{m^2}\right)^{s+1/2}\frac{\Gamma(s+1)}{\Gamma(s)}\sum_{a=1}^N \zeta_{K_{0aa}}(s+1)\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, U_{aa}(x)\end{aligned}$$

Zeta functions and Kink masses

$$\triangle M_K = \lim_{s \to -1/2} \left(\triangle E_K^C(s) + \triangle E_K^R(s) \right)$$

Outline

$\lambda(\phi^4)_2$ Kinks

- The action

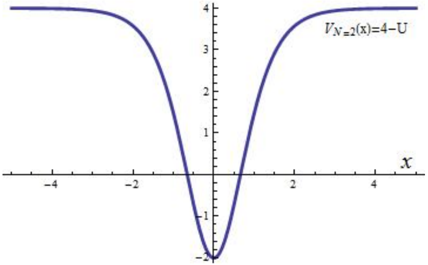
$$S[\phi] = \frac{m^2}{2\lambda}\int d^2x \left\{ \frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} - (\phi^2(t,x)-1)^2 \right\} \quad , \quad \sigma_{11} = \sigma_1^2 = 0$$

- Vacua, Kinks and Hessians

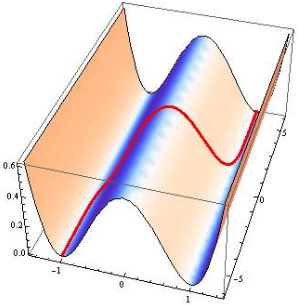
$$\phi^V = \pm 1 \quad , \quad K_0 = -\frac{d^2}{dx^2} + 4$$

$$\phi^K(x) = \pm\tanh(x - x_0) \quad , \quad K = -\frac{d^2}{dx^2} + 4 - \frac{6}{\cosh^2 x}$$

Kink valleys and well



Kink potential well



Kink mountain pass

Kink spectral heat/zeta functions

- Kink heat function

$$\mathrm{Tr} e^{-\beta K} = \frac{e^{-4\beta}}{\sqrt{4\pi}} \left(\frac{l}{\sqrt{\beta}} + \sqrt{4\pi}(e^{\beta}\mathrm{Erf}[\sqrt{\beta}] + e^{4\beta}\mathrm{Erf}[2\sqrt{\beta}]) \right)$$

- Kink zeta function

$$\zeta_K(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta\, \beta^{s-1} \left(\mathrm{Tr} e^{-\beta K} - 1 \right)$$

$$= \frac{1}{\sqrt{4\pi}\Gamma(s)} \left[\frac{l}{4^s}\Gamma(s-\frac{1}{2}) + \left(\frac{2}{3^{s+\frac{1}{2}}}\, {}_2F_1[\frac{1}{2}, \frac{1}{2}+s, \frac{3}{2}; -\frac{1}{3}] - \frac{1}{4^s s} \right) \Gamma(s+\frac{1}{2}) \right]$$

- One-loop mass shift

$$\triangle M_K = \lim_{s\to -1/2} \left(\triangle E_K^C(s) + \triangle E_K^R(s) \right) = \left(\frac{1}{2\sqrt{6}} - \frac{3}{\pi\sqrt{2}} \right) \hbar m = -0.471113\hbar m$$

- Nuclear Physics **B635**, 525 (2002)

Divergence cancelation

- Pole in the Kink Casimir energy

$$\begin{aligned}
\Delta E_K^C &= \frac{\hbar m}{2\sqrt{2\pi}} \lim_{\varepsilon\to 0} \left(\frac{2\mu^2}{m^2}\right)^{\varepsilon} \frac{\Gamma(\varepsilon)}{\Gamma(-\frac{1}{2}+\varepsilon)} \left[\frac{2}{3^\varepsilon}\, {}_2F_1[\tfrac{1}{2},\varepsilon,\tfrac{3}{2},-\tfrac{1}{3}] - \frac{1}{(-\frac{1}{2}+\varepsilon)\,4^{-\frac{1}{2}+\varepsilon}}\right] \\
&= \frac{\hbar m}{2\sqrt{2\pi}} \lim_{\varepsilon\to 0} \left[-\frac{3}{\varepsilon} + 2 + \ln\frac{3}{4} - 3\ln\frac{2\mu^2}{m^2} - {}_2F_1^{(0,1,0,0)}[\tfrac{1}{2},0,\tfrac{3}{2},-\tfrac{1}{3}] + o(\varepsilon)\right] \\
&= -\frac{\hbar m}{2\sqrt{2\pi}} \lim_{\varepsilon\to 0} \left[\frac{3}{\varepsilon} + 3\ln\frac{2\mu^2}{m^2} - \frac{\pi}{\sqrt{3}}\right]
\end{aligned}$$

- Mass renormalization Kink energy

$$\begin{aligned}
\Delta E_K^R &= -\frac{3\hbar m}{\sqrt{2\pi}} \lim_{\varepsilon\to 0} \left(\frac{2\mu^2}{m^2}\right)^{\varepsilon} \frac{4^{-\varepsilon}\Gamma(\varepsilon)}{\Gamma(-\frac{1}{2}+\varepsilon)} \\
&= \frac{3\hbar m}{2\sqrt{2\pi}} \lim_{\varepsilon\to 0} \left[\frac{1}{\varepsilon} + \ln\frac{2\mu^2}{m^2} - \ln 4 + (\psi(1) - \psi(-\frac{1}{2})) + o(\varepsilon)\right] \\
&= \frac{\hbar m}{2\sqrt{2\pi}} \lim_{\varepsilon\to 0} \left[\frac{3}{\varepsilon} + 3\ln\frac{2\mu^2}{m^2} - 2(2+1)\right]
\end{aligned}$$

Outline

BNRT model

- Parameters

$$\sigma_1^2 = 0 \;, \quad \sigma_2^2 = 1 - \frac{\sigma}{2} \;, \quad \sigma_{11} = \frac{3}{2} \;, \quad \sigma_{22} = \frac{\sigma^2 - 1}{2} \;, \quad 1 + \sigma_{12} = 2\sigma(\sigma + 1)$$

- The BNRT action

$$\begin{aligned} S[\phi_1, \phi_2] &= \frac{m^2}{\lambda} \int d^2x \left\{ \frac{1}{2} \left(\frac{\partial \phi_1}{\partial x^\mu} \frac{\partial \phi_1}{\partial x_\mu} + \frac{\partial \phi_2}{\partial x^\mu} \frac{\partial \phi_2}{\partial x_\mu} \right) - \frac{1}{2} (\phi_1^2(t,x) + \phi_2^2(t,x) - 1)^2 \right. \\ &- \left. \frac{3}{2} \phi_1^4 + \frac{1 - \sigma^2}{2} \phi_2^4 + (1 - 2\sigma(\sigma + 1)) \phi_1^2 \phi_2^2 + \frac{\sigma - 2}{2} \phi_2^2 \right\} \end{aligned}$$

- “Vertical" sector

$$\begin{aligned} \phi_1^{V^{(2)}} &= 0 \;, \quad \phi_2^{V^{(2)}} = \pm \frac{1}{\sqrt{2\sigma}} \;, \quad V(0, \pm \frac{1}{\sqrt{2\sigma}}) = \frac{3}{8} \\ \phi_1^K(x) &= 0 \;, \quad \phi_2^K(x) = \pm \frac{1}{\sqrt{2\sigma}} \tanh[\sqrt{\frac{\sigma}{2}}(x - x_0)] \end{aligned}$$

BNRT kink fluctuations

- "Horizontal" kinks

$$\phi_1^{V^{(1)}} = \pm\frac{1}{2} \ , \ \phi_2^{V^{(1)}} = 0 \quad , \quad V(\pm\frac{1}{2},0) = \frac{3}{8}$$

$$\phi_1^K(x) = \pm\frac{1}{2}\tanh(x-x_0) \ , \ \phi_2^K(x) = 0$$

$$\mu_1^2 = 4(1-\sigma_1^2) = 4 \qquad , \qquad \mu_2^2 = 2\left(\frac{1-\sigma_1^2}{1+2\sigma_{11}}(1+\sigma_{12}) - (1-\sigma_2^2)\right) = \sigma^2$$

$$K_0 = \begin{pmatrix} -\frac{d^2}{dx^2}+4 & 0 \\ 0 & -\frac{d^2}{dx^2}+\sigma^2 \end{pmatrix}$$

$$K = \begin{pmatrix} -\frac{d^2}{dx^2}+4-\frac{6}{\cosh^2 x} & 0 \\ 0 & -\frac{d^2}{dx^2}+\sigma^2-\frac{\sigma(\sigma+1)}{\cosh^2 x} \end{pmatrix}$$

Aristocratic TK1 Kinks

- Transparent Pösch-Teller potentials: $\sigma = N \in \mathbb{Z}^+$

$$K_{22} = -\frac{d^2}{dx^2} + N^2 - \frac{N(N+1)}{\cosh^2 x} \quad , \quad N \in \mathbb{Z}^+ \quad , \quad A = N + \frac{1}{2} \; , \; \frac{4\mu_a^2}{\mu_c^2} = N^2$$

- Discrete spectrum

$$\varepsilon_2^2(j) = (2N-j)j \quad , \quad j = 0, 1, 2, \cdots, N$$

$$f_0^2(x) = \frac{1}{\cosh^N x} \; , \quad f_j^2(x) = \Pi_{r=0}^{j-1}\left(-\frac{d}{dx} + (N-r)\text{tanh}x\right)\frac{1}{\cosh^{N-j} x} \; , \quad j \geq 1$$

- Scattering states

$$\varepsilon_2^2(k) = k^2 + N^2 \quad , \quad f_k^2(x) = \Pi_{p=1}^{N}\left(-\frac{d}{dx} + p\text{tanh}x\right) e^{ikx}$$

$$\delta_2(k) = 2\text{arctan}\left[\frac{\text{Im}(\Pi_{p=1}^N (p-ik))}{\text{Re}(\Pi_{p=1}^N (p-ik))}\right] \quad , \quad \rho_2(k) = \frac{l}{2\pi} - \frac{1}{\pi}\sum_{p=1}^{N}\frac{p}{p^2+k^2}$$

Aristocratic TK1 Kinks

$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=1$
0	0	0	0	0	0	0	0	0	0
1	3	5	7	9	11	13	15	17	19
	4	8	12	16	20	24	28	32	36
		9	15	21	27	33	39	45	51
			16	24	32	40	48	56	64
				25	35	45	55	65	75
					36	48	60	72	84
						49	63	77	91
							64	80	96
								81	99
									100

Kink heat and zeta function

- The heat function

$$\begin{aligned} \mathrm{Tr}_{L^2} e^{-\beta K} &= \sum_{j=0}^{N} e^{-\beta(2N-j)} + \frac{l}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-\beta(k^2+N^2)}\left[1-\frac{2}{l}\sum_{p=1}^{N}\frac{p}{p^2+k^2}\right] \\ &= \frac{e^{-\beta N^2}}{\sqrt{4\pi}}\left(\frac{l}{\sqrt{\beta}}+\sqrt{4\pi}\sum_{p=1}^{N} e^{\beta p^2}\mathrm{Erf}[p\sqrt{\beta}]\right) \end{aligned}$$

- The zeta function

$$\begin{aligned} \zeta_K^*(s) &= \frac{1}{\Gamma(s)}\int_0^{\infty} d\beta\, \beta^{s-1}\left(\mathrm{Tr}_{L^2} e^{-\beta K}-1\right) \\ &= \frac{1}{\sqrt{\pi}}\left[\frac{l}{2}\frac{\Gamma(s-\frac{1}{2})}{N^{2s-1}\Gamma(s)}+\left(\sum_{j=1}^{N-1}\frac{2j}{(N^2-j^2)^{s+\frac{1}{2}}}\cdot {}_2F_1[\frac{1}{2},\frac{1}{2}+s,\frac{3}{2};-\frac{j^2}{N^2-j^2}]-\right.\right. \\ &\qquad \left.\left.-\frac{1}{N^2 s}\right)\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}\right] \end{aligned}$$

- Nuclear Physics **B681**, 163 (2004)

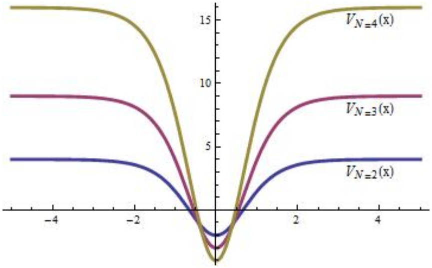
Aristocratic TK1 Kinks

- The $N=3$ and $N=4$ BNRT TK1 kinks

$$\zeta_K^*(s) = \frac{1}{\sqrt{\pi}}\left[\frac{l}{2}\frac{\Gamma(s-\frac{1}{2})}{3^{2s-1}\Gamma(s)} + \left(\frac{2}{8^{s+\frac{1}{2}}}\cdot {}_2F_1[\frac{1}{2},\frac{1}{2}+s,\frac{3}{2};-\frac{1}{8}] + \right.\right.$$

$$\left.\left. + \frac{4}{5^{s+\frac{1}{2}}}\cdot {}_2F_1[\frac{1}{2},\frac{1}{2}+s,\frac{3}{2};-\frac{4}{5}] - \frac{1}{9s}\right)\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}\right]$$

$$\triangle E_K^{N=3} = -(0.47113+0.766861)\hbar m \quad , \quad \triangle E_K^{N=4} = -(0.47113+1.52420)\hbar m$$



Outline

1. Deformed $\mathbb{O}(N)$ linear sigma models
   - Vacuum fluctuations
   - Kink fluctuations
   - One-loop kink mass shift
   - N=1: the $\lambda(\phi^4)_2$ model
   - N=2: the Bazeia-Nascimento-Ribeiro-Toledo model
   - N=3: the massive non-linear $\mathbb{S}^2$-sigma model

$\mathbb{S}^2$-sigma model action

- Formal $\lambda \to \infty$ limit

$$S[\chi_1,\chi_2,\chi_3] = \frac{1}{2}\int dx^2 \left\{\sum_{a=1}^{3}\frac{\partial\chi_a}{\partial x_\mu}\cdot\frac{\partial\chi_a}{\partial x^\mu} - \alpha_1^2\chi_1^2(t,x) - \alpha_2^2\chi_2^2(t,x) - \alpha_3^2\chi_3^2(t,x)\right\}$$

$$\chi_1^2(t,x)+\chi_2^2(t,x)+\chi_3^3(t,x) = R^2 \quad ; \quad R^2 = \frac{m^2}{\lambda}$$

$$\alpha_1^2 = 2\sigma_1^2 \quad , \quad \alpha_2^2 = 2\sigma_2^2 \quad , \quad \alpha_3^2 = 2\sigma_3^2 \; , \quad 0 < \sigma^2 = \frac{\sigma_2^2-\sigma_3^2}{\sigma_1^2-\sigma_3^2} < 1$$

$$S = \frac{1}{2}\int dtdx\left\{\partial_\mu\chi_1\partial^\mu\chi_1 + \partial_\mu\chi_2\partial^\mu\chi_2 + \frac{(\chi_1\partial_\mu\chi_1+\chi_2\partial_\mu\chi_2)^2}{R^2-\chi_1^2-\chi_2^2} - \right.$$
$$\left. - \quad \chi_1^2(t,x) - \sigma^2\cdot\chi_2^2(t,x)\right\}$$

- Interactions come from geometry

$$\frac{(\chi_1\partial_\mu\chi_1+\chi_2\partial_\mu\chi_2)(\chi_1\partial^\mu\chi_1+\chi_2\partial^\mu\chi_2)}{R^2-\chi_1^2-\chi_2^2} \simeq$$
$$\simeq \frac{1}{R^2}\left(1+\frac{1}{R^2}(\chi_1^2+\chi_2^2)+\frac{1}{R^4}(\chi_1^2+\chi_2)^2+\cdots\right)\cdot(\chi_1\partial_\mu\chi_1+\chi_2\partial_\mu\chi_2)\left(\chi_1\partial^\mu\chi_1+\chi_2\partial^\mu\chi_2\right)$$

$\mathbb{S}^2$-kinks

- Spherical coordinates and field equations

$$\chi_1(t,x) = R\sin\theta(t,x)\cos\varphi(t,x)$$

$$\chi_2(t,x) = R\sin\theta(t,x)\sin\varphi(t,x) \ , \quad \partial^\mu(\sin^2\theta\partial_\mu\varphi) - \frac{1}{2}\bar{\sigma}^2\sin^2\theta\sin 2\varphi = 0$$

$$\chi_3(t,x) = R\cos\theta(t,x) \ , \quad \Box\theta - \frac{1}{2}\sin 2\theta\left(\partial^\mu\varphi\partial_\mu\varphi - \cos^2\varphi - \sigma^2\sin^2\varphi\right) = 0$$

- $K1$ kinks

$$\varphi_{K_1}(x) = \frac{\pi}{2} \ , \ \varphi_{K_1^*}(x) = \frac{3\pi}{2} \quad ; \quad \theta_{K_1}(x) = 2\arctan e^{\pm\sigma(x-x_0)}$$

$$\chi_1^{K_1}(x) = 0 \ , \quad \chi_2^{K_1}(x) = \pm\frac{R}{\cosh[\sigma(x-x_0)]} \ , \quad \chi_3^{K_1}(x) = \pm R\tanh[\sigma(x-x_0)]$$

- $K2$ kinks

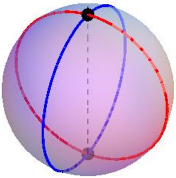
$$\varphi_{K_2}(x) = 0 \ , \ \varphi_{K_1^*}(x) = \pi \quad ; \quad \theta_{K_2}(x) = 2\arctan e^{\pm(x-x_0)}$$

$$\chi_1^{K_2}(x) = \pm\frac{R}{\cosh[(x-x_0)]} \ , \quad \chi_2^{K_2}(x) = 0 \ , \quad \chi_3^{K_2}(x) = \pm R\tanh[(x-x_0)]$$

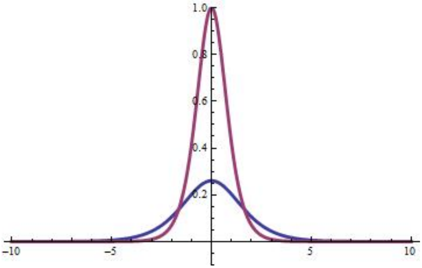
$\mathbb{S}^2$-kink energies

- Topological $\mathbb{S}^2$-kink energies

$$E^C_{K_1} = m\sqrt{2}R^2\sigma \qquad , \qquad E^C_{K_2} = m\sqrt{2}R^2$$



K_1 and K_2 orbits



K_1 and K_2 energy densities

- Physical Review Letters **101**, 131602 (2008)

K_1 one-loop mass shifts

- Geodesic deviation plus Hessian of the potential for the K_1 kink (in a parallel frame to the kink orbit)

$$K = \begin{pmatrix} -\frac{d^2}{dx^2} + \sigma^2 - \frac{2\sigma^2}{\cosh^2 \sigma x} & 0 \\ 0 & -\frac{d^2}{dx^2} + 1 - \frac{2\sigma^2}{\cosh^2 \sigma x} \end{pmatrix}$$

- Cahill-Comtet-Glauber formula

$$\begin{aligned} E^{OL}_{K_1}(\sigma) &= E^{C}_{K_1}(\sigma) + \Delta M_{K_1}(\sigma) + \mathcal{O}(\frac{\hbar^2}{R^2}) \; ; \quad \nu_1 = \arccos(0) = \frac{\pi}{2} \quad , \quad \nu_2 = \arccos \bar{\sigma} \\ &= \sqrt{2} m R^2 \sigma - \frac{\hbar m \sigma}{\sqrt{2}\pi}[\sin \nu_1 + \frac{1}{\sigma} \sin \nu_2 - \nu_1 \cos \nu_1 - \frac{1}{\sigma} \nu_2 \cos \nu_2] + \mathcal{O}(\frac{\hbar}{R^2}) \end{aligned}$$

- K_1 one-loop mass shift

$$\begin{aligned} E^{OL}_{K_1}(\sigma) &= \sqrt{2} m R^2 \sigma - \frac{\hbar m \sigma}{\sqrt{2}\pi} \left[2 - \frac{\bar{\sigma}}{\sigma} \arccos(\bar{\sigma}) \right] + \mathcal{O}(\frac{\hbar^2}{R^2}) \\ E^{OL}_{K_1}(\frac{1}{2}) &= \frac{m}{\sqrt{2}} R^2 - \frac{3\hbar m}{2\sqrt{2}\pi} \left(\frac{2}{3} - \frac{\pi}{6\sqrt{3}} \right) + \mathcal{O}(\frac{\hbar^2}{R^2}) \end{aligned}$$

- Physical Review **D 79**, 121503 (2009)

K_2 one-loop mass shifts

- Geodesic deviation plus Hessian of the potential for the K_2 kink (in a parallel frame to the kink orbit)

$$K = \begin{pmatrix} -\frac{d^2}{dx^2} + 1 - \frac{2}{\cosh^2 x} & 0 \\ 0 & -\frac{d^2}{dx^2} + \sigma^2 - \frac{2}{\cosh^2 x} \end{pmatrix}$$

- Cahill-Comtet-Glauber formula

$$\begin{aligned} E^{OL}_{K_2}(\sigma) &= E^{C}_{K_2}(\sigma) + \Delta M_{K_2}(\sigma) + \mathcal{O}(\frac{\hbar^2}{R^2}) \; ; \quad \nu_1 = \arccos(0) = \frac{\pi}{2} \; , \; \nu_2 = \arccos(i\bar{\sigma}) \\ &= \sqrt{2} m R^2 - \frac{\hbar m \sigma}{\sqrt{2}\pi}[\sin\nu_1 + \frac{1}{\sigma}\sin\nu_2 - \nu_1\cos\nu_1 - \frac{1}{\sigma}\nu_2\cos\nu_2] + \mathcal{O}(\frac{\hbar}{R^2}) \end{aligned}$$

- K_2 one-loop mass shift

$$\begin{aligned} E^{OL}_{K_2}(\sigma) &= \sqrt{2} m R^2 - \frac{\hbar m \sigma}{\sqrt{2}\pi}\left[\frac{1}{\sigma} + \sqrt{2-\sigma^2} - i\frac{\pi}{2}\bar{\sigma} + \bar{\sigma}\log\left(\sqrt{2-\sigma^2} - \bar{\sigma}\right)\right] \\ &+ \mathcal{O}(\frac{\hbar^2}{R^2}) \end{aligned}$$

$$E^{OL}_{K_2}(\frac{1}{2}) = \frac{m}{\sqrt{2}}R^2 - \frac{\hbar m}{4\sqrt{2}\pi}\left(4 + \sqrt{7} + \sqrt{3}\log\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right) - i\frac{\pi\sqrt{3}}{2}\right) + \mathcal{O}(\frac{\hbar^2}{R^2})$$