

Description of Physical Systems

States \times Observables



Probability measures on \mathbb{R}

$$f \times \mathcal{G} \rightarrow \{\mu\}$$

$$\mu(g, A)(E)$$

probability to find
a result in E
when we measure A
while the system
is in the state g

Remark
Cartesian property of maps

$$\text{Map}(M, N; P) \cong \text{Map}(M, \text{Map}(N; P)) = \text{Map}(N, \text{Map}(M, P))$$

.. Duality

$$M \leftrightarrow \mathcal{F} \quad \text{ex: } \mathcal{F} \times M \rightarrow P \quad (f, m) \mapsto f(m)$$

$$M = \text{Hom}_{\mathbb{R}}(\mathcal{F}, \mathbb{R})$$

product structure on \mathcal{F}
"differential structure" on M

Description of Quantum Systems

Schrödinger picture

Hilbert space \mathcal{H}

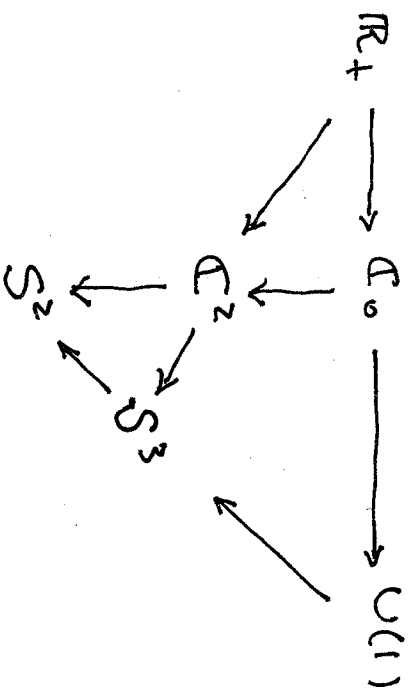
$\mathbb{C}_0 \rightarrow \mathcal{H} \rightarrow \mathbb{R}\mathcal{H}$

States

Observables: Real (Un)Bounded operators on \mathcal{H}

Example

$$\mathcal{H} = \mathbb{C}^2$$



Observables

2×2 Hermitian matrices $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$

Heisenberg picture

\mathcal{A} \mathbb{C}^* -algebra

Observables real elements of \mathcal{A} , $\text{Re } \mathcal{A}$

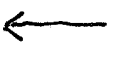
States (normalized positive) functionals on \mathcal{A}

$$\rho(\mathbb{1}) = 1 \quad \rho(A^*A) \geq 0$$

$$f \in (\text{Re } \mathcal{A})^* = \mathbb{C}^*$$

$$\mathbb{C} \hookrightarrow \mathcal{F}(\mathcal{P}) \quad A \mapsto \hat{A} \quad \text{such that} \quad \hat{A}(\rho) = \rho(A)$$

$$\mathcal{A} \otimes \mathcal{A} \quad A \otimes B \rightarrow \hat{A} \cdot \hat{B} \quad (\hat{A} \cdot \hat{B})(\rho) = \hat{A}(\rho) \hat{B}(\rho)$$



$$\hat{A} * \hat{B} \quad (\hat{A} * \hat{B})(\rho) = \rho(AB)$$

A familiar example

\mathfrak{g} a Lie algebra

$$\mathfrak{g} = \text{Lin}(\mathfrak{g}^*, \mathbb{R}) \subset \mathcal{F}(\mathfrak{g}^*)$$

$$(\hat{u} \cdot \hat{v})(\alpha) = \hat{u}(\alpha) \hat{v}(\alpha) \quad \hat{v}, \hat{u} \in \mathcal{F}(\mathfrak{g}^*)$$

$$\widehat{[u, v]}(\alpha) = \{ \hat{u}, \hat{v} \}(\alpha) \quad \text{Poisson bracket}$$

Tensors

$$[\wedge(d\hat{u}, d\hat{v})](\alpha) = \alpha([u, v])$$

Poisson tensor on linear functions
extended by setting

$$\wedge(\hat{v} \cdot d\hat{u}) = \hat{v} \wedge (d\hat{u})$$

From algebraic structures to geometrical structures

\mathcal{A} algebra

$$m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$$

$$m(A, B) \in \mathcal{A}$$

$$S: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \quad (A, B) \mapsto \frac{1}{2} (m(A, B) + m(B, A))$$

$$w: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \quad (A, B) \mapsto \frac{1}{2} (m(A, B) - m(B, A))$$

$$S + w = m$$

on $\mathcal{A}^* = \text{Lin}_{\mathbb{K}}(\mathcal{A}, \mathbb{K})$

$$G(d\hat{A}, d\hat{B})(\rho) = \rho(S(A, B))$$

$$\wedge(d\hat{A}, d\hat{B})(\rho) = \rho(w(A, B))$$

• \mathcal{A} \mathbb{C}^* -algebra

v) Jordan structure

w) Lie structure

u) Lie-Jordan structure

Given a Lie-Jordan structure, is it possible to define a \mathbb{C}^* -algebra?

In $(\text{Re } \mathcal{A})$ $S : (A, B) \rightarrow S(A, B)$ is Jordan

$\tilde{w} : (A, B) \rightarrow -i w(A, B)$ is Lie

$\tilde{w}(A)$ is a derivation for the Jordan product
 $m = S + a \tilde{w}$

becomes an associative product for a selected value of the parameter a

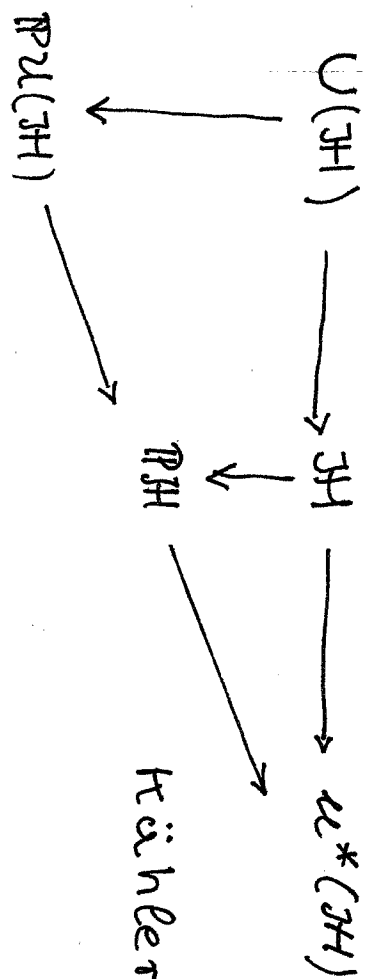
Remark

In geometrical terms

$$\wedge (d\tilde{A}, d[\Gamma G(d\tilde{B}, d\tilde{C})]) = G(\wedge(d\tilde{A}, d\tilde{B}), d\tilde{C}) + G(d\tilde{B}, d\wedge(d\tilde{A}, d\tilde{C}))$$

is a very strong requirement

From Schrödinger to Heisenberg



The symmetric and the skew symmetric part of the Hermitian tensor field on \mathfrak{H} projects onto a symmetric and skew symmetric contravariant tensor on $\mathcal{U}^*(\mathfrak{H})$

$\mathcal{U}^*(\mathfrak{H})$ Stratified Kählerian manifold

From Heisenberg to Schrödinger

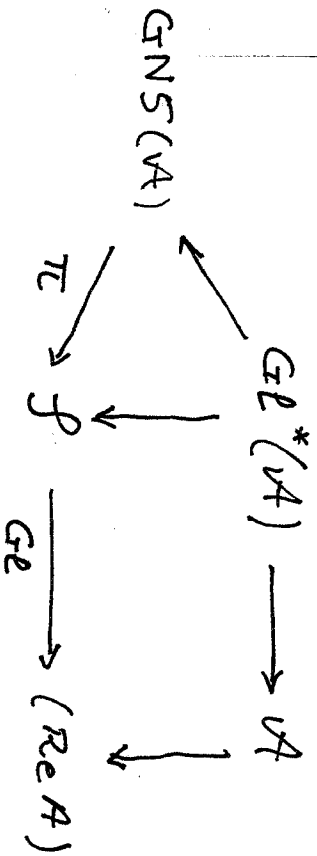
$$\mathcal{A} \cong T^*(\mathcal{U}^*(\mathcal{H})) \cong T^*[\overline{(\operatorname{Re} \mathcal{A})^*}] \cong (\operatorname{Re} \mathcal{A})^* \oplus (\operatorname{Re} \mathcal{A})$$

States $\xrightarrow{G_E} \operatorname{Re} \mathcal{A}$

Gleason's theorem

$$f(A) = \operatorname{Tr}(\rho \cdot A)$$

GNS-bundle



Fibers of the GNS-bundle

$$\pi^{-1}(\rho) = \{A \cdot \rho\} \cong \mathcal{A}_\rho$$

$$(A \cdot \rho)(B) = f(A^*B)$$

Example

2-level system

$$A = \begin{pmatrix} z_1 & x_1 \\ z_2 & x_2 \end{pmatrix}$$

$$\rho = \frac{1}{2} \begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix}$$

$$\text{Tr } \rho = 1$$

$$x^2 + y^2 + z^2 \leq 1$$

select $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$A \cdot \rho = \begin{pmatrix} z_1 & 0 \\ z_2 & 0 \end{pmatrix}$$

$$\rho(A^* B) = \langle A | B \rangle_\rho = \langle A \rho | B \rho \rangle = \bar{z}_1 r_1 + \bar{z}_2 r_2$$

$$B = \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix}$$

Fibers of the GNS-bundle \mathbb{C}^2 and $\mathbb{C}^2 \times \mathbb{C}^2$
 ρ stratified by the rank of ρ

PJM minimal orbit of $\mathcal{U}(\mathcal{H})$ in ρ
 extremal states of ρ
 irreducible representations of \mathcal{U}

GNs - construction

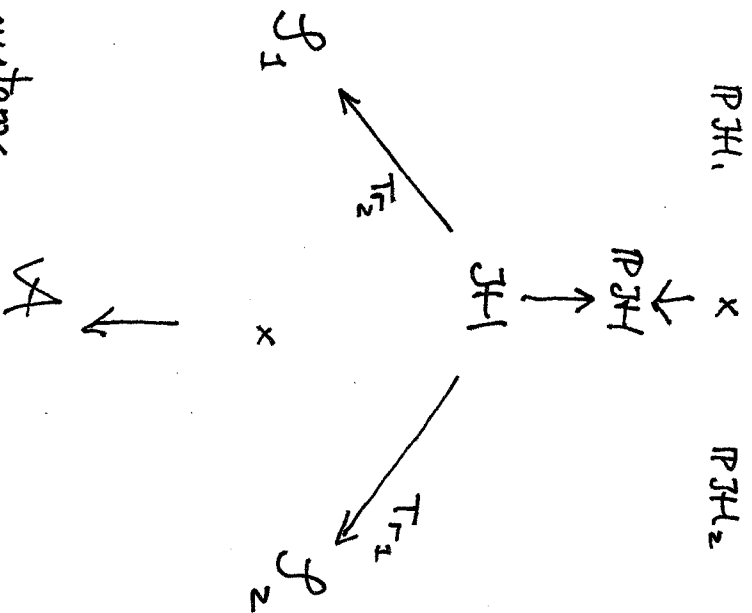
construction of a Kählerian momentum map
a canonical realization of a Kählerian
stratified space

A different version of Wigner's
theorem for Quantum Symmetries

Composite systems

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$$



- Evolution of open systems
- measurement problem
- uniqueness of \mathcal{H}