Description of Physical Systems

$$\text{States} \times \text{Observables}$$
$$\downarrow$$
$$\text{Probability measures on } \mathbb{R}$$

$\mathcal{S} \times \mathcal{O} \to \{\mu\}$

$\mu(\rho, A)(E)$ probability to find a result in E when we measure A while the system is in the state ρ

Remark
Cartesian property of maps

. $\mathrm{Map}(M, N; P) \rightleftarrows \mathrm{Map}(M, \mathrm{Map}(N; P)) = \mathrm{Map}(N, \mathrm{Map}(M, P))$

.. Duality

$M \leftrightarrow \mathcal{F}$ $\quad e_v : \mathcal{F} \times M \to P \quad (f, m) \mapsto f(m)$

$M = \mathrm{Hom}_{\mathbb{R}}(\mathcal{F}, \mathbb{R})$ $\quad$ product structure on $\mathcal{F}$
"differential structure" on M

Description of Quantum Systems

Schrödinger picture

Hilbert space $\mathcal{H}$ $\qquad \mathbb{C}_0 \longrightarrow \mathcal{H} \longrightarrow \mathbb{P}\mathcal{H}$ $\qquad$ States

Observables : Real (Un) Bounded operators on $\mathcal{H}$

Example $\quad \mathcal{H} = \mathbb{C}^2$

$$\begin{tikzcd}
\mathbb{R}_+ \arrow[r] \arrow[dr] & \mathbb{C}_0 \arrow[r] \arrow[d] & U(1) \arrow[d] \\
& \mathbb{C}^2 \arrow[r] \arrow[d] & S^3 \arrow[dl] \\
& S^2 &
\end{tikzcd}$$

Observables
  2×2 Hermitian matrices $\quad (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$

Heisenberg picture

$\mathcal{A}$ $\mathbb{C}^*$ - algebra

Observables real elements of $\mathcal{A}$, $\mathrm{Re}\,\mathcal{A}$

States (normalized positive) functionals on $\mathcal{A}$

$$\rho(\mathbb{1}) = 1 \qquad \rho(A^*A) \geq 0$$

$$\mathcal{S} \subset (\mathrm{Re}\,\mathcal{A})^* = \mathcal{O}^*$$

$\mathcal{O} \hookrightarrow \mathcal{F}(\mathcal{S})$ $\quad A \mapsto \hat{A}$ such that $\quad \hat{A}(\rho) = \rho(A)$

$$\begin{array}{lll} \mathcal{A} \otimes \mathcal{A} & A \otimes B \longrightarrow \hat{A} \cdot \hat{B} & (\hat{A} \cdot \hat{B})(\rho) = \hat{A}(\rho)\, \hat{B}(\rho) \\ & \quad\downarrow & \\ & \hat{A} * \hat{B} & (\hat{A} * \hat{B})(\rho) = \rho(AB) \end{array}$$

A familiar example

$\mathfrak{g}$ a Lie algebra $\qquad \mathfrak{g} = \mathrm{Lin}(\mathfrak{g}^*, \mathbb{R}) \subset \mathcal{F}(\mathfrak{g}^*)$

$(\hat{u} \cdot \hat{v})(\alpha) = \hat{u}(\alpha)\,\hat{v}(\alpha) \qquad \hat{v}, \hat{u} \in \mathcal{F}(\mathfrak{g}^*)$

$\widehat{[u, v]}(\alpha) = \{\hat{u}, \hat{v}\}(\alpha)$ Poisson bracket

Tensors

$[\Lambda(d\hat{u}, d\hat{v})](\alpha) = \alpha([u, v])$

Poisson tenson on linear functions
extended by setting

$\Lambda(\hat{v} \cdot d\hat{u}) = \hat{v}\,\Lambda(d\hat{u})$

From algebraic structures to geometrical structures

$\mathcal{A}$ algebra $\qquad m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad m(A,B) \in \mathcal{A}$

$s : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad (A,B) \mapsto \frac{1}{2}\left(m(A,B) + m(B,A)\right)$

$\omega : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad (A,B) \mapsto \frac{1}{2}\left(m(A,B) - m(B,A)\right)$

$s + \omega = m$

On $\mathcal{A}^* = \mathrm{Lin}_{\mathbb{K}}(\mathcal{A}, \mathbb{K})$

$G(d\hat{A}, d\hat{B})(\rho) = \rho(s(A,B))$

$\Lambda(d\hat{A}, d\hat{B})(\rho) = \rho(\omega(A,B))$

- $\mathcal{A}$ C^*- algebra

i) Jordan structure

ii) Lie structure

iii) Lie- Jordan structure

Given a Lie-Jordan structure, is it possible to define a $\mathbb{C}^*$-algebra ?

In $(\mathrm{Re}\,\mathcal{A})$ $\quad s : (A,B) \to s(A,B)$ is Jordan

$\tilde{\omega} : (A,B) \longrightarrow -i\,\omega(A,B)$ is Lie

$\tilde{\omega}(A)$ is a derivation for the Jordan product

$$m = s + a\,\tilde{\omega}$$

becomes an associative product for a selected value of the parameter a

Remark
In geometrical terms

$$\Lambda\left(d\hat{A}, d\left[G(d\hat{B}, d\hat{C})\right]\right) = G\left(d\Lambda(d\hat{A}, d\hat{B}), d\hat{C}\right) + G\left(d\hat{B}, d\Lambda(d\hat{A}, d\hat{C})\right)$$

is a very strong requirement

From Schrödinger to Heisenberg

$$\begin{tikzcd} U(\mathcal{H}) \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & \mathfrak{u}^*(\mathcal{H}) \\ & \mathbb{P}\mathcal{H} \arrow[ur] & \\ \mathbb{P}\mathcal{U}(\mathcal{H}) \arrow[ur] & & \end{tikzcd}$$

Kählerian momentum map

The symmetric and the skew symmetric part of the Hermitian tensor field on $\mathcal{H}$ projects onto a symmetric and skew symmetric contravariant tensor on $\mathfrak{u}^*(\mathcal{H})$

$\mathfrak{u}^*(\mathcal{H})$ Stratified Kählerian manifold

From Heisenberg to Schrödinger

$$\mathcal{A} \rightleftarrows T^*(\mathcal{U}^*(\mathcal{H})) \equiv T^*\left[(\mathrm{Re}\,A)^*\right] \equiv (\mathrm{Re}\,\mathcal{A})^* \oplus (\mathrm{Re}\,\mathcal{A})$$

$$\text{States} \xrightarrow{Gl} \mathrm{Re}\,\mathcal{A}$$

Gleason's theorem

$$\rho(A) = Tr(\hat{\rho}\cdot A)$$

GNS - bundle

$$\begin{tikzcd}
 & Gl^*(\mathcal{A}) \arrow[dl] \arrow[dd] \arrow[r] & \mathcal{A} \arrow[dd] \\
GNS(\mathcal{A}) \arrow[dr, "\pi"'] & & \\
 & \rho \arrow[r, "Gl"'] & (\mathrm{Re}\,A)
\end{tikzcd}$$

Fibers of the GNS-bundle $\quad \pi^{-1}(\rho) = \{A\cdot\rho\} \equiv \mathcal{A}_\rho$

$$(A\cdot\rho)(B) = \rho(A^*B)$$

Example

2-level system

$$A = \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix} \qquad \rho = \frac{1}{2}\begin{vmatrix} 1+x & y+iz \\ y-iz & 1-x \end{vmatrix} \qquad \begin{array}{l} \mathrm{Tr}\,\rho = 1 \\ x^2+y^2+z^2 \leq 1 \end{array}$$

select $\rho = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$

$$A \cdot \rho = \begin{vmatrix} z_1 & 0 \\ z_2 & 0 \end{vmatrix} \qquad \rho(A^* B) = \langle A | B \rangle_\rho = \langle A\rho | B\rho \rangle$$

$$= \bar{z}_1 r_1 + \bar{z}_2 r_2$$

$$B = \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix}$$

Fibers of the GNS-bundle $\mathbb{C}^2$ and $\mathbb{C}^2 \times \mathbb{C}^2$

$\mathcal{S}$ stratified by the rank of ρ

$\mathbb{P}\mathcal{H}$ minimal orbit of $\mathcal{U}(\mathcal{H})$ in $\mathcal{S}$

extremal states of $\mathcal{S}$

irreducible representations of $\mathcal{A}$

GNS-construction

construction of a kählerian momentum map

a canonical realization of a kählerian stratified space

A different version of Wigner's theorem for Quantum Symmetries

Composite systems

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ $\qquad\qquad$ $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$

$$\begin{tikzcd}
\mathbb{P}\mathcal{H}_1 & \times \arrow[d] & \mathbb{P}\mathcal{H}_2 \\
& \mathbb{P}\mathcal{H} & \\
& \mathcal{H} \arrow[u] \arrow[dl, "\mathrm{Tr}_2"'] \arrow[dr, "\mathrm{Tr}_1"] & \\
\mathcal{S}_1 & \times \arrow[d] & \mathcal{S}_2 \\
& \mathcal{A} &
\end{tikzcd}$$

- Evolution of open systems
- measurement problem
- uniqueness of $\hbar$