Spin-dipole Effects in Noncommutative QM: A Supersymmetric Example.

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The problem

$$[x_i, x_j] = i\theta^2 \epsilon_{ijk} s^k$$

Outline

- Motivation
- The algebra
- Standard NCQM vs. Alternative NCQM
  - Space Quantization
  - Symmetries
  - Interactions
- Two examples:
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  - The supersymmetric harmonic oscillator
- Spontaneous rotational symmetry breaking
- Conclusions and Questions

Motivation

Original motivation: Snyder[1],

$$[x_\mu, x_\nu] = i\theta^2 \mathcal{M}_{\mu\nu}$$

$\mathcal{M}$ are the Lorentz generators.
Nonrelativistic case,

$$[x_i, x_j] = i\theta^2 \epsilon_{ijk} J_k$$

or[2]

$$[x_i, x_j] = i\theta^2 \epsilon_{ijk} L_k$$

or

$$[x_i, x_j] = i\theta^2 \epsilon_{ijk} s_k$$

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[1] H.S.Snyder, *Phys. Rev.* **71** (1947)
[2] M. Battisti and S. Meljanac, arXiv:08123755v3 [hep − th]

The algebra

$$[x_i, x_j] = i\theta^2 \epsilon_{ijk} s^k$$

The rest of the algebra satisfying the Jacobi identity,

$$[s_i, s_j] = i\epsilon_{ijk} s^k$$

$$[x_i, s_j] = i\theta\epsilon_{ijk} s^k$$

$$[x_i, p_j] = i\delta_{ij},$$

Realizations

- Standard NCQM,

$$[x_i, x_j] = i\theta_{ij}$$

can be realized by Bopp's shift; the theory can be written in terms of commutative varables:

$$x_i \longrightarrow x_i + \frac{i}{2}\theta_{ij}\partial_j$$

- It is possible to realize the new commutators in terms of a commutative variable x,

$$x_i \longrightarrow \mathrm{x}_i = x_i + \theta s_i$$

with,

$$[x_i, x_j] = [x_i, s_j] = 0$$

Space Quantization

- Space satisifies uncertainity relation,

$$\Delta x_i \Delta x_j \geq \theta_{ij}/2$$

and it is quantized,

$$r_{ij}^2 = x_i^2 + x_j^2 = 2\theta_{ij}\left(N + \frac{1}{2}\right)$$

- In our case, we have uncertainity relation, but the space is not quantized, becouse the commutative coordinates x_i have continious spectrum.

$$\mathrm{x}_i = \underset{\text{cont}}{x_i} + \underset{\text{disc}}{s_i}$$

Symmetries

In standard NCQM we loose rotation and PT symmetries,

- Rotation:

$$\mathcal{R}[x_i, x_j]\mathcal{R}^{-1} = R_{ik}R_{jm}[x_k, x_m] = R_{ik}R_{jm}\theta_{km}$$

- $\mathcal{PT}$: Time reversal is an antilinear operator $\mathcal{T}$,

$$\mathcal{PT}[x_i, x_j](\mathcal{PT})^{-1} = [x_i, x_j] = i\theta_{ij}$$

On the other hand,

$$\mathcal{PT}i\theta_{ij}(\mathcal{PT})^{-1} = -i\theta_{ij}$$

Symmetries

- Under rotations, $[x_i, x_j]$ and $\epsilon_{ijk}S^k$, transform as antisymmetric tensors, and therefore the relation,

$$[x_i, x_j] = i\theta\epsilon_{ijk}S^k$$

  preserves rotation symmetry.
- $\mathcal{PT}$ symmetry is preserved by

$$[x_i, x_j] = i\theta^2\epsilon_{ijk}s_k$$

Interactions

- Standrad case: Some local interactions become, in general, nonlocal operators. A central potential will be,

$$V(r^2) \to V\left(r^2 - i\theta_{ij}x_i\partial_j - \frac{1}{4}\theta_{ik}\theta_{jm}\partial_i\partial_j\right)$$

- Our case: local interactions are always local in the NC case. In particular for a central potential,

$$V(r^2) \to V_{\text{even}}(r^2 + 3\theta^2/4 + \theta r) + \theta S_r V_{\text{odd}}(r^2 + 3\theta^2/4 + \theta r)$$

where,

$$V_{\text{even/odd}} \equiv \frac{1}{2}[V(r^2 + 3\theta^2/4 + \theta r) \pm V(r^2 + 3\theta^2/4 - \theta r)]$$

and

$$S_r = \frac{\vec{x} \cdot \vec{S}}{r}$$

The harmonic oscillatror

Noncoommutative harmonic oscillator,

$$H_{NC} = \frac{1}{2}\vec{p}^2 + \frac{1}{2}\omega^2\vec{\mathbf{x}}^2$$

is equivalent to the commutative model,

$$H = \frac{1}{2}\vec{p}^2 + \frac{1}{2}\omega^2\vec{x}^2 + \frac{3}{8}(\omega\theta)^2 + \omega^2\theta\vec{x}\cdot\vec{s}$$

We have two coupled second order differential equations.

A non perturbative property

In a semiclassical approach, an effective hamiltonian is

$$H^{\pm}_{\mathrm{eff}} = \frac{1}{2}(\vec{p} - \vec{\mathcal{A}}^{\pm})^2 + \frac{1}{2}\vec{x}^2$$

where,

$$\vec{\mathcal{A}}^{\pm} = i\langle \pm(x(t))|\vec{\nabla}| \pm (x(t))\rangle$$

and

$$\vec{x}(t) \cdot \vec{S}| \pm (x(t))\rangle = \pm r(t)| \pm (x(t))\rangle$$

These potentials correspond to a monopole placed at the origin with strength $\mp 1/2$[1],

$$\mathcal{F}^{\pm}_{ij} = \pm\frac{1}{2}\epsilon_{ijk}\frac{x_k}{r^3}$$

[1]M.V.Berry. Proc. R. Lond. **A392** (1984).

The supersymmetric harmonic oscillator

The supersymmetric version of the HO is,

$$H = \frac{\omega}{2}\left\{\mathcal{S}^\dagger, \mathcal{S}\right\}$$

The Hilbert space is,

$$\mathcal{H} = L^2(\mathbb{R}^3) \times \mathbb{C}^2_{\text{spin}} \times \mathbb{C}^2$$

where the first two factors correspond to the space wave function and spin space, and the last one is an auxiliary space. If we decompose the Hilbert space in two sectors,

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

$$\mathcal{H}^+ = L^2(\mathbb{R}^3) \times \mathbb{C}_{\text{spin}} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathcal{H}^- = L^2(\mathbb{R}^3) \times \mathbb{C}_{\text{spin}} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

And the supercharges take an state from one sector to the other one,

$$\mathcal{S}\,\mathcal{H}^+ \subset \mathcal{H}^-, \qquad \mathcal{S}\,\mathcal{H}^- \subset \mathcal{H}^+$$

SUSY HO

The supercharges are,

$$\mathcal{S} = Q \otimes \sigma^-, \qquad \mathcal{S}^\dagger = Q^\dagger \otimes \sigma^+$$

$$Q = Q_i s^i = \left(\frac{1}{\sqrt{2\omega}} \partial_i + \sqrt{\frac{\omega}{2}} \hat{\mathrm{x}}_i \right) s^i$$

$$Q\dagger = Q_i^\dagger s^i = \left(-\frac{1}{\sqrt{2\omega}} \partial_i + \sqrt{\frac{\omega}{2}} \hat{\mathrm{x}}_i \right) s^i$$

These operators hold the relations,

$$[Q_i, Q_j^\dagger] = \delta_{ij} + \frac{i}{2} \omega \theta^2 \epsilon_{ijk} s^k$$

$$[Q_i, Q_j] = [Q_i^\dagger, Q_j^\dagger] = \frac{i}{2} \omega \theta^2 \epsilon_{ijk} s^k$$

These operators tend to the standard a_i and $a^\dagger$ when θ goes to zero.

SUSY HO

Hence, the hamiltonian is,

$$H_s = \left(-\frac{1}{2}\vec{\nabla}^2 + \frac{1}{2}\omega^2\vec{x}^2 + \omega\theta\vec{x}\cdot\vec{s} + \frac{9}{8}\theta^2\right) \otimes \mathbb{I}_{2\times 2}$$

$$- \left(2\vec{L}\cdot\vec{s} + \frac{3}{2}\right) \otimes \sigma^3$$

Projected on $\mathcal{H}^{\pm}$,

$$H_s^{\pm} = \left(-\frac{1}{2}\vec{\nabla}^2 + \frac{1}{2}\omega^2\vec{x}^2 + \omega\theta\vec{x}\cdot\vec{s} + \frac{9}{8}\theta^2\right) \mp \left(2\vec{L}\cdot\vec{s} + \frac{3}{2}\right)$$

The fundamental state

Supersymmetry,

$$[\mathcal{S}, H_s] = [\mathcal{S}^\dagger, H_s] = 0,$$

and then, for the fundamental state,

$$\mathcal{S}|\Omega\rangle = \mathcal{S}^\dagger|\Omega\rangle = 0$$

That means that,

$$Q \otimes \sigma^-|\Omega\rangle = Q^\dagger \otimes \sigma^+|\Omega\rangle = 0$$

with,

$$Q_i = a_i + \theta\omega^{\frac{1}{2}} s_i$$

$$Q_i^\dagger = a_i^\dagger + \theta\omega^{\frac{1}{2}} s_i$$

The fundamental state

Two possibilities:

- The first one gives the conditions,

$$|\Omega\rangle = \begin{pmatrix} 0 \\ \Psi(x) \end{pmatrix} \in \mathcal{H}^-$$

  and,

$$Q^\dagger \Psi(x) = (a_i^\dagger + \omega^{\frac{1}{2}} \theta s_i) s^i \Psi(x) = 0$$

  This yields no normalizable stetes.

- The other possibility is,

$$|\Omega\rangle = \begin{pmatrix} \Psi(x) \\ 0 \end{pmatrix} \in \mathcal{H}^+$$

  and

$$Q \Psi(x) = (a_i + \omega^{\frac{1}{2}} \theta s_i) s^i \Psi(x) = 0$$

The fundamental state

This implies,

$$\left(\vec{a}\cdot\vec{s}+\frac{3}{4}\omega^{\frac{1}{2}}\theta\right)\Psi(x)=0$$

The solutions for this equation has the form,

$$|\Omega(\vec{z})\rangle=\mathcal{N}(|z|)\chi(z1,z_2,z_3)\mathrm{exp}(\omega^{\frac{1}{2}}z_i a_i^\dagger)|0\rangle\otimes\left(\begin{array}{c}1\\0\end{array}\right)$$

where z_1, z_2 and z_3 are complex parameters, $\mathcal{N}=e^{\omega\vec{z}^*\cdot\vec{z}}$ is a normalization constat and χ is a normalized spinor such that,

$$\left(\vec{z}\cdot\vec{s}+\frac{3}{4}\theta\right)\chi(z)=0$$

Monopole and θ quantization(?).

The fundamental state

The condition $\det\left(\vec{z}\cdot\vec{s}+\frac{3}{4}\theta\right)=0$ implies,

$$z^2 = \frac{9}{4}\theta^2$$

If we write

$$z_i = u_i + iv_i,$$

the condition $z^2 = 9\theta^2/4$ means that,

$$\vec{u}^2 - \vec{v}^2 = \frac{9}{4}\theta^2$$

$$\vec{u}\cdot\vec{v} = 0$$

All different Vacua live in a four-dimonsional space invariant under $SO_{\mathbb{C}}(3)$.

Spontaneous rotational symmetry breaking

These vacua are not invariant under rotations. In fact, the generators,

$$J_i = \epsilon_{ijk} x^j p^k + s_I = i\epsilon_{ijk} a_j^\dagger a_k + s_i$$

do not annihilate the vacua,

$$J_i |\Omega(\vec{z})\rangle \neq 0$$

for $z \neq 0$.
The VEV of the different operators are,

$$\langle \vec{x} \rangle_z = \sqrt{2}\vec{u}$$

$$\langle \vec{p} \rangle_z = \sqrt{2}\omega\vec{v}$$

$$\langle \vec{s} \rangle_Z = \frac{3}{4}\frac{\theta}{u^2}\vec{u} + \frac{1}{2u^2}\vec{u} \times \vec{v}$$

where $\langle \vec{s} \rangle^2 = 1/4$ and the minimum value of u is $u_{\min} = 3\theta/2$.

Conclusions and Questions

- Are rotational symmetry breaking and θ quantization general features of this algebra?
- Is it possible to generalize the model to any spin representatcion.
- Many particle interactions would induce electric multipole-spin intereacting terms (some ultra-cooled cadmium gases[1])
- Next step is considering $[x_i, x_j] = i\epsilon_{ijk}J_k$.

[1] J.Werner et al. Phys. Rev. Lett. **94** 183201