WESS-ZUMINO-WITTEN and CHERN-SIMONS THEORIES for NON-SIMPLY-CONNECTED GROUPS

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"The time we are able to spend together is too precious to be passed in telling stories"

from **Jan Potocki**'s "The manuscript found in Saragossa"

The interplay between **field theory** and **geometry** and **topology** was one of the main trends of theoretical physics of the last decades

The topic of the talk: revisiting relations between two well known examples of field theories involving geometry and topology:

- **Wess-Zumino-Witten model**
- **Chern-Simons theory**

- **WZW model**: a 2D conformal sigma model with Lie a group G target and coupling constant k
  - if G is compact, simple and simply connected then k has to be a positive integer $1, 2, \ldots$
  - if G is not simply connected then k is further restricted

- **CS theory**: a 3D topological gauge theory with the gauge group G and coupling constant k
  - if G is compact, simple and simply connected then k has to be a positive integer $1, 2, \ldots$
  - if G is not simply connected then k is even more constraint

Example: for $G = SO(3)$, k has to be even for **WZW** and multiple of 4 for **CS**

Reason: topological obstructions to definition of **Feynman** amplitudes

Earlier works on the case of non-simply-connected groups and related issues:

- **Abouelsaood-Gepner**, Ṗhys. Lett. B 176 (1986), 380
- **Felder-Kupiainen-G.**, Commun. Math. Phys. 117 (1988), 127
- **Dijkgraaf-Witten**, Commun. Math. Phys. 129 (1990), 393
- **Carey-Johnson-Murray-Stevenson-Wang**, Commun. Math. Phys. 259 (2005), 577
- **Waldorf**, "Multiplicative bundle gerbes with connection" arXiv:0804.4835

Based on joint work with **Rafal Suszek** and **Konrad Waldorf** that put the old results of **F.-K.-G.** from 1988 in a new light

Main idea: **Feynman** amplitudes receive contributions from

higher Abelian holonomies

Needed:

- classification of geometric structures behind such holonomies:
  - (**bundle-**)**n-gerbes** (**with connection**) and their **modules**
  - **multiplicative** and **equivariant structures** on them, etc.
- classification of obstructions to their existence

Here: only a flavor of a particular consideration applying to the relation between **WZW** and **CS**

- standard **Abelian holonomy**: the case of electromagnetic field
  - $\varphi : S^1 \to M$,
  - A Abelian gauge field - a 1-form,
  - $dA = F$ gauge field strength - a 2-form

$$\exp\left[2\pi i \int_{S^1} \varphi^* A\right] \;=\; Hol_{\mathcal{L}}(\varphi)$$

↖ **line bundle** on M

The **RHS** makes sense even if only the closed 2-form F exists globally and has integral periods

- **Example**: a magnetic monopole with the quantized charge

Relevant geometric structure: **line bundle with connection** $\mathcal{L}$,
F is the curvature of $\mathcal{L}$

- 1 degree higher **Abelian holonomy**: the case of **Kalb-Ramond** field
  - $\varphi : \Sigma \to M$, Σ oriented closed surface
  - B **Kalb-Ramond** field - a 2-form,
  - $dB = H$ torsion 3-form

$$\exp\left[2\pi i \int_\Sigma \varphi^* B\right] \;=\; Hol_{\mathcal{G}}(\varphi)$$

$\nwarrow$ **gerbe** on M

The **RHS** makes sense even if only the closed 3-form H exists globally and has integral periods

- **Example**: 3-form $H_k = \frac{k}{24\pi^2}\,\mathrm{tr}\,(g^{-1}dg)^3$ on simple compact Lie group G

Relevant geometric structure: **gerbes with connection** $\mathcal{G}$,

H is the curvature of $\mathcal{G}$

- **WZW** model uses the holonomy of a gerbe $\mathcal{G}_k$ on group G with 3-form curvature H_k to define the **Wess-Zumino** contribution to the **Feynman** amplitudes of the classical field $g : \Sigma \to G$:

$$\exp\Big[iS_{WZ}(g)\Big] = Hol_{\mathcal{G}_k}(g)$$

- **Main property**: for a **homotopy** $\tilde{g} : [0,1] \times \Sigma \mapsto G$

$$Hol_{\mathcal{G}_k}(\tilde{g}(1,\cdot)) = Hol_{\mathcal{G}_k}(\tilde{g}(0,\cdot)) \exp\Big[\int_{[0,1]\times\Sigma} \tilde{g}^* H_k\Big]$$

- $\mathcal{G}_k$ exists iff the periods of H_k are integers
  $\Rightarrow$ selection rules for the **level** k (**Felder-Kupiainen-G.** 1988)

- 2 degrees higher **Abelian holonomy**: the case of 3-form field H
    - $\varphi : S \to M$, S oriented closed 3-manifold
    - H 3-form field,
    - $dH = P$ 4-form field strength

$$\exp\left[2\pi i \int_S \varphi^* H\right] = Hol_{\mathcal{K}}(\varphi)$$

↖ **2-gerbe** on M

The **RHS** makes sense even if only the closed 4-form P exists globally and has integral periods

- **Example**: the **Pontryagin** 4-form $P_k(\boldsymbol{A}) = \frac{k}{8\pi^2}\,\mathrm{tr}\,\boldsymbol{F}(\boldsymbol{A})^2$ on M corresponding to a (non-abelian) group-G gauge field $\boldsymbol{A}$

Relevant geometric structure: **2-gerbes with connection** $\mathcal{K}$,
P is the curvature of $\mathcal{K}$

- **CS theory** uses the holonomy of a 2-gerbe $\mathcal{K}_k(\boldsymbol{A})$ with the **Pontryagin** curvature $P_k(\boldsymbol{A})$ for a non-Abelian gauge field $\boldsymbol{A}$ to define the **Feynman** amplitudes for the maps $\varphi : N \to M$

$$\exp\left[2\pi i \int_N \varphi^* CS_k(\boldsymbol{A})\right] \;=\; Hol_{\mathcal{K}_k(\boldsymbol{A})}(\varphi)$$

where the **CS** forms $CS_k(\boldsymbol{A}) = \frac{k}{8\pi^2}\,\mathrm{tr}\,(\boldsymbol{A} d\boldsymbol{A} + \frac{2}{3}\boldsymbol{A}^3)$ satisfy $d\,CS_k(\boldsymbol{A}) = P_k(\boldsymbol{A})$ but, in general, are defined only locally

- Geometric relation between to the **WZW** and **CS** theory:

  a gerbe $\mathcal{G}_k$ (on G) equipped with a **multiplicative structure** induces canonically a 2-gerbe $\mathcal{K}_k(\boldsymbol{A})$ on M for each gauge-field $\boldsymbol{A}$

Multiplicative structure on $\mathcal{G}_k$ = good behavior of $\mathcal{G}_k$ under group multiplication

- One has:

$$H_k(gg') \;=\; H_k(g) \;+\; H_k(g') \;+\; d\,\rho_k(g,g')$$

where $\rho_k(g,g') = \frac{k}{8\pi^2}\,\mathrm{tr}\,(g^{-1}dg)(g'dg'^{-1})$ is a 2-form on G^2
but, in general, for $g,\, g' : \Sigma \to G$,

$$Hol_{\mathcal{G}_k}(gg') \;=\; c(g,g') \cdot Hol_{\mathcal{G}_k}(g)\; Hol_{\mathcal{G}_k}(g')\; \exp\Big[2\pi i \int_\Sigma \rho_k(g,g')\Big]$$

with a phase $c(g,g')$ (the "**Polyakov-Wiegmann** formula")

- **Thm**. A **multiplicative structure** on $\mathcal{G}_k$ exists iff $c(g,g') \equiv 1$ **!!!**

- **2-cocycle property**:

$$c(g,g')\; c(gg',g'') \;=\; c(g',g'')\; c(g,g'g'')$$

- The case of G simply-connected:

  $c(g,g') \equiv 1$ and a (unique) multiplicative structure always exists on $\mathcal{G}_k$ for $k = 1, 2, \ldots$ (original **Polyak.-Wieg.** formula)

- Remains the case of G non-simply-connected $G = \tilde{G}/Z$ with $Z \subset Z(\tilde{G})$ $\leftarrow$ center of simply-connected group $\tilde{G}$

  For compact simple groups $\tilde{G}$:

$$\begin{array}{llll}
Z(\tilde{G}) = \mathbf{Z}_n & \text{for} & \tilde{G} = SU(n) \\
Z(\tilde{G}) = \mathbf{Z}_2 & \text{for} & \tilde{G} = Spin(2n+1),\; Sp(2n),\; E_7 \\
Z(\tilde{G}) = \mathbf{Z}_3 & \text{for} & \tilde{G} = E_6 \\
Z(\tilde{G}) = \mathbf{Z}_4 & \text{for} & \tilde{G} = Spin(4n+2) \\
Z(\tilde{G}) = \mathbf{Z}_2^2 & \text{for} & \tilde{G} = Spin(4n) \\
Z(\tilde{G}) = 0 & \text{for} & \tilde{G} = E_8,\; F_4,\; G_2
\end{array}$$

A bit more on gerbes with **multiplicative structures**

$$m : G \times G \longrightarrow G \qquad pr_1 : G \times G \to G, \quad pr_2 : G \times G \to G$$

- **Gerbes** on M form a tensor 2-category with pullbacks
- **multiplicative structure** on $\mathcal{G}_k$ is given by

  (a). 1-morphisms $\alpha : m * \mathcal{G}_k \longrightarrow pr_1^* \mathcal{G}_k \otimes pr_2^* \mathcal{G}_k \otimes \mathcal{I}_{\rho_k}$
  where $\mathcal{I}_{\rho_k}$ is a trivial gerbe with the holonomy given by $2\pi i \times$ the integral of ρ_k

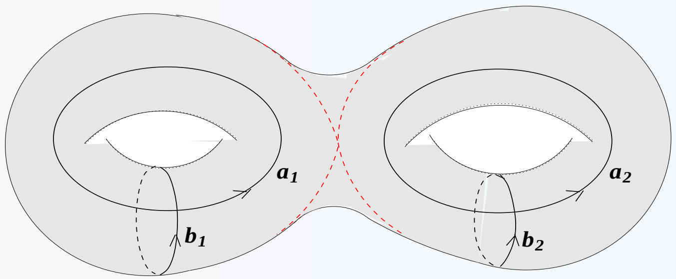
  (b). associativity 2-morphisms satisfying a pentagon identity between compositions of 1-morphisms α over $G \times G \times G$,

- **Obstructions**
  - to (a) are naturally cohomology classes in $H^2(Z^2, U(1))$ not just 2-cocycles **!!!**
  - to (b) appear to vanish

Further properties of $c(g,g')$:

- $c(g,g')$ depends only on the **winding "numbers"** of g, g':

$$z_{2\alpha-1} = P\,\mathrm{e}^{i\int_{a_\alpha} g^{-1}dg}\,, \quad z_{2\alpha} = P\,\mathrm{e}^{i\int_{b_\alpha} g^{-1}dg} \quad \in \; Z$$



$\alpha = 1, \ldots, 2\gamma$ where γ is the **genus** of Σ, and the same for g'

- **Factorization** in terms od genus 1 data:

$$c(z_1, \ldots z_{2\gamma}; z'_1, \ldots z'_{2\gamma})$$
$$= \; c(z_1, z_2; z'_1, z'_2) \;\; \cdots \;\; c(z_{2\gamma-1}, z_{2\gamma}; z'_{2\gamma-1}, z'_{2\gamma})$$

$\Rightarrow$ **Multiplicative structure** on $\mathcal{G}_k$ exists iff the 2-**cocycle** on Z^2

$$c(z_1, z_2; z_1', z_2') \equiv 1$$

Calculation of $c(z_1, z_2; z_1', z_2')$ (**Felder-Kupiainen-G.** 1988):

- for $Z = \mathbf{Z}_n$ with the generator $\zeta = \mathrm{e}^{2\pi i\theta}$, θ in the **Cartan** algebra

  For fields $g_{q_1,q_2}(\mathrm{e}^{i\varphi^1}, \mathrm{e}^{i\varphi^2}) = \mathrm{e}^{i(q_1\varphi^1 + q_2\varphi^2)\theta}$ on $S^1 \times S^1$,
  $Hol_{\mathcal{G}_k}(g_{q_1,q_2}) = 1$ for dimensional reasons hence:

$$c(\zeta^{q_1}, \zeta^{q_2}; \zeta^{q_1'}, \zeta^{q_2'}) \equiv c(g_{q_1,q_2}, g_{q_1',q_2'})$$
$$= \exp\Big[-2\pi i \int\limits_{S^1\times S^1} \rho_k(g_{q_1,q_2}, g_{q_1',q_2'})\Big] = \mathrm{e}^{-\pi i k(q_1 q_2' - q_2 q_1')\,\mathrm{tr}\,\theta^2}$$

  and $c \equiv 1$ iff $\frac{k}{2}\,\mathrm{tr}\,\theta^2 \in \mathbf{Z}$ (for $SO(3)$, $\theta = \frac{1}{2}\sigma^3$ hence $\frac{k}{4} \in \mathbf{Z}$)

- for $Z = \mathbf{Z}_2^2$ with generators $\zeta^a = \mathrm{e}^{2\pi i \theta^a}$, $a = 1, 2$ (for $Spin(4n)$)

  With notation $\zeta^{\vec{q}} \equiv \mathrm{e}^{2\pi i (\vec{q}\cdot\vec{\theta})}$,

$$c(\zeta^{\vec{q}_1}, \zeta^{\vec{q}_2}; \zeta^{\vec{q}'_1}, \zeta^{\vec{q}'_2}) = \left(\pm\, \mathrm{e}^{\frac{\pi i k}{2}}\right)^{\vec{q}_1\times\vec{q}'_2 - \vec{q}_2\times\vec{q}'_1} \cdot \mathrm{e}^{-\pi i k\, \mathrm{tr}\left((\vec{q}_1\cdot\vec{\theta})(\vec{q}'_2\cdot\vec{\theta}) - (\vec{q}'_1\cdot\vec{\theta})(\vec{q}_2\cdot\vec{\theta})\right)}$$

The obstruction to **multiplicative structure** were hidden in other contexts in 3 equivalent forms:

- **2-cocycle** $c(z_1, z_2; z'_1, z'_2)$ on Z^2

- **bihomomorphisms** $\xi_c \in Hom(Z \otimes Z, U(1))$
  (**Kreuzer−Schellekens** 1994)

$$\xi_c(z_1 \otimes z'_2) = c(z_1, 1; 1, z'_2)$$

- **cohomology class** $[\chi_c] \in H^2(Z^2, U(1))$

$$\chi_c(z_1, z_2; z'_1, z'_2) = \xi_c(z_1 \otimes z'_2)$$

Conclusions, applications, prospects

- A **multiplicative structure** on the gerbe $\mathcal{G}_k$ on Lie group G determines canonically:
    - **2-gerbe** $\mathcal{K}_k(\boldsymbol{A})$ on M for each G-gauge field $\boldsymbol{A}$ on M
    - a **central extension** of the loop group LG (an extended symmetry of the **WZW** model) - (**Waldorf** 2008)
    - **partition functions** of the corresponding **WZW** and **CS** models
    - a symmetric **bibrane** (a defect or a discrete symmetry of the **WZW** theory) - (**Fuchs-Schweigert-Waldorf** 2007)
- Similar structures appear in **gauged sigma models** in **Kalb-Ramond** backgrounds, e.g. in the study of global aspects of the **coset models** or of the **T-duality**. These are the **equivariant structures** on **gerbes**

Advertisement

Gerbes introduced for the first time by **Jean Giraud** in 1971 seem still vowed for bright future as providing clues to topological and geometric effects in **field theory**

Learn **gerbes** !

A bit of history of gerbes:

• Giraud	1971	abstract gerbes
• Deligne, Beilinson	mid 80's	Deligne cohomology
• Alvarez	1985	cohom.struct. in WZactions
• G.	1987	Deligne coh. in topological field theory & WZW
• Brylinski	1991	loop spaces characteristic classes & geometric quantization
• Murray, Murray-Stevenson	1994 & 1999	bundle gerbes
• Carey-Mickelsson-Murray	1995	gerbes, anomalies & index
• Chatterjee-Hitchin Brylinski Meinrenken G.-Reis	1998 2000 2002 2000	} (bdle)gerbes on Lie groups
• Schreiber-Schweigert-Waldorf	2005	gerbes & orientifolds
• G.-Suszek-Waldorf	2007	WZW orientifolds via gerbes
• Belov-Hull-Minasian	2007	gerbes & $T-$duality
• Hull-Lindström-Roček -von Unge-Zabzine	2008	gerbes and generalized geom.

- **Local presentation of a gerbe** $\mathcal{G}$ with curvature H on manifold M with local covering (O_a):

  a collection (B_a, A_{ab}, g_{abc}) of local 2-forms, 1-forms, and $U(1)$-valued functions such that

$$\begin{aligned} dB_a &= H\,, \\ B_b - B_a &= dA_{ab}\,, \\ A_{ab} - A_{ac} + A_{bc} &= i\, d\ln g_{abc} \\ g_{abc}\, g_{abd}^{-1}\, g_{acd}\, g_{bcd}^{-1} &= 1 \end{aligned}$$

  modulo $(d\Pi_a, \Pi_b - \Pi_a - id\,\ln\chi_{ab}, \chi_{ab}^{-1}\chi_{ac}\chi_{bc}^{-1})$

- **Holonomy** $Hol_{\mathcal{G}}(\varphi)$ for $\varphi : \Sigma \to M$:

  $\varphi^*(B_a, A_{ab}, g_{abc}) \sim (B, 0, 1)$ where B is global on Σ

$$Hol_{\mathcal{G}}(\varphi) = \exp\left[2\pi i \int_\Sigma B\right]$$

- **Local story for 2-gerbes** by analogy