

# Massive 3d gravity and Brown-Henneaux diffeomorphisms

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work (in progress) in collaboration with Stefan Theisen

## TOC

- ▶ Three dimensional New Massive gravity
- ▶ Old-New Massive gravity (bigravity)
- ▶ 4d motivations for bigravity.
- ▶ Back to three dimensions. Asymptotics and Conformal Fields

- ▶ **TMG** (Topologically massive gravity) has received great attention recently. Li, Song and Strominger (2008) found a chiral point of TMG with interesting properties.
- ▶ **New Massive Gravity.** Bergshoeff, Holm and Townsend (2009) have constructed an extension of TMG for a massive graviton in three dimensions (omitting the Chern-Simons term):

$$I[g] = \int d^3x \sqrt{g} \left( \sigma R + \frac{1}{m^2} \left( R^{\mu\nu} R_{\mu\nu} - \frac{3}{8} R^2 \right) - 2\lambda m^2 \right).$$

Despite being quadratic in the curvatures, this action has second order field equations. Linearization yields the Pauli-Fierz action for a massive spin-2 particle in three dimensions.

The trick: Add another field  $f_{\mu\nu}$  and write

$$I[g, f] = \int \sqrt{g} \left( R - 2\lambda m^2 + f^{\mu\nu} G_{\mu\nu} - \frac{m^2}{4} (f^{\mu\nu} f_{\mu\nu} - f^2) \right)$$

$f_{\mu\nu}$  is non-dynamical and the actions are the same. Linearizing

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + ah_{\mu\nu} + bk_{\mu\nu}, \quad f_{\mu\nu} = \bar{g}_{\mu\nu} + ch_{\mu\nu} + ck_{\mu\nu}.$$

$a, b, c, d$  can be chosen such that the fluctuations  $h_{\mu\nu}$  and  $k_{\mu\nu}$  are decoupled. The quadratic action is

$$I[h, k] = \int h^{\mu\nu} \mathcal{G}(h)_{\mu\nu} + k^{\mu\nu} \mathcal{G}(k)_{\mu\nu} - \frac{m^2}{4} (k^{\mu\nu} k_{\mu\nu} - k^2)$$

(trivial in 3d)                      (massive spin-2 particle)

$$h^{\mu\nu} \mathcal{G}(h)_{\mu\nu} \equiv \frac{-1}{4} D_\mu h_{\nu\rho} D^\mu h^{\nu\rho} + \frac{1}{2} D_\lambda h_{\mu\nu} D^\mu h^{\lambda\nu} + \frac{1}{4} D_\mu h D^\mu h - \frac{1}{2} D_\mu h D_\nu h^{\mu\nu} + \frac{1}{2l^2} \left( h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right)$$

- ▶ **'Old-New' massive gravity.** A similar structure appears in a different way, proposed almost 40 years (!) ago by Isham, Salam and Strathdee. Consider an action for *two* spin-2 fields

$$I[g_{\mu\nu}, f_{\mu\nu}] = \frac{1}{16\pi G} \int \sqrt{g}(R_g - \Lambda) + \sigma \sqrt{f}(R_f - \Lambda) + m^2 \sqrt{g}(g_{\mu\nu} - f_{\mu\nu})(g_{\alpha\beta} - f_{\alpha\beta})(f^{\mu\alpha} f^{\nu\beta} - f^{\mu\nu} f^{\alpha\beta})$$

Again the action for fluctuations can be diagonalized

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + ah_{\mu\nu} + bk_{\mu\nu}, \quad f_{\mu\nu} = \bar{g}_{\mu\nu} + ch_{\mu\nu} + dk_{\mu\nu}.$$

obtaining the same linear action as Bergshoeff, Hohm, Townsend

$$I[h, k] = \int h^{\mu\nu} \mathcal{G}(h)_{\mu\nu} \quad + \quad k^{\mu\nu} \mathcal{G}(k)_{\mu\nu} - \frac{m^2}{4} (k^{\mu\nu} k_{\mu\nu} - k^2)$$

(trivial in 3d)                      (massive spin-2 particle)

**Why multigravity?** Maxwell theory is extended into Yang-Mills theory:

$$\int F_{\mu\nu} F^{\mu\nu} \rightarrow \int F_{1\mu\nu} F_1^{\mu\nu} + F_{2\mu\nu} F_2^{\mu\nu} + \dots + \mathbf{interactions}$$

This extension has deep applications (in Physics and Mathematics). It is extremely tempting to do the same with the spin two field:

$$\int \sqrt{g}(R - 2\Lambda) \rightarrow \int \sqrt{g_1}(R_1 - 2\Lambda_1) + \sqrt{g_2}(R_2 - 2\Lambda_2) + \dots \\ + \mathbf{interactions}$$

However, for spin 2, the interactions do not have the same nice structure:

## The $N$ -Gravity action functional

$$I[g_{\mu\nu}^a] = \sum_{a=1}^N \int \sqrt{g_a} R_a \quad (1)$$

is invariant under independent transformations on each metric

$$\delta g_{\mu\nu}^a = \xi_{a,\mu}^\alpha g_{\alpha\nu}^a + \xi_{a,\nu}^\alpha g_{\mu\alpha}^a + \xi_a^\alpha g_{\mu\nu,\alpha}^a, \quad a = 1 \dots N. \quad (2)$$

Is this symmetry (or a deformation of it) preserved by interactions?  
For example,

$$g_1^{\mu\nu} g_{2\mu\nu} \quad (3)$$

is invariant *provided*  $\xi_1^\mu = \xi_2^\mu$ . A general perturbative theorem has been proved. For weak fields,

$$g_{\mu\nu}^a = \eta_{\mu\nu} + h_{\mu\nu}^a \quad (4)$$

no cross potential  $V(h_{\mu\nu}^a)$  preserving the full  $N$  dimensional symmetry exists (Boulanger-Damour-Gualtieri-Henneaux (2000)).

Interactions of the Isham, Salam, Strathdee family ( $u + v = \frac{1}{2}$ )

$$V = \det(g)^u \det(g)^v \left( f^{\mu\nu} g_{\mu\nu} + \kappa \left( (f^{\mu\nu} g_{\mu\nu})^2 - f^{\mu\nu} g_{\nu\alpha} f^{\alpha\beta} g_{\beta\mu} \right) \right)$$

preserve the diagonal subgroup.

- ▶ If there are two metrics, different matter fields may couple to different metrics. This implies that while standard particles follow geodesics of  $g_{\mu\nu}$ , other exotic particles may follow geodesics of  $f_{\mu\nu}$

$$\frac{d^2 y^\mu}{ds^2} + \Gamma(f)^\mu_{\alpha\beta} \frac{dy^\alpha}{ds} \frac{dy^\beta}{ds} = 0$$

Wouldn't it be interesting?

- ▶ See Damour, Kogan; Arkani-Hamed, Georgi, Schwartz; Blas, Deffayet, Garriga; Berezhiani, Comelli, Nesti, Pilo; Blas, Comelli, Nesti, Pilo;... for more work on this.



**Isham, Storey** (1978) found the exact Schwarzschild (type II) metric for the Isham-Salam-Strathdee potential,

$$ds^2 = - \left( 1 - \frac{2M}{r} + \Lambda_1 r^2 \right)^2 dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \Lambda_1 r^2} + r^2 d\Omega^2$$
$$df^2 = - \left( 1 - \frac{2\mu}{r} + \Lambda_2 r^2 \right)^2 dt'^2 + \frac{dr^2}{1 - \frac{2\mu}{r} + \Lambda_2 r^2} + r^2 d\Omega^2$$

where

$$t' = t + \text{'function'}(r)$$

- ▶  $\Lambda_1$  and  $\Lambda_2$  are integration constants not related to action couplings. Asymptotically AdS spaces coexist with de-Sitter cosmologies as solutions to the same action!
- ▶ Thermodynamics? Charges? (are finite); Entropy? To be done.

**Cosmology:** The Friedmann equation for the ansatz

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2, \quad (5)$$

$$df^2 = -X(t)^2 dt^2 + Y(t)^2 d\vec{x}^2 \quad (6)$$

is

$$\frac{\dot{a}^2}{a^2} = \frac{Y^3/X}{a^3} + \frac{\Omega_{baryons}}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda \quad (7)$$

A remarkable property of this class of theories is that near  $a = 0$   $Y^3/X$  is constant. Thus we can set

$$\frac{Y^3}{X} = \Omega_{dark\ matter}$$

and the contribution from  $f_{\mu\nu}$  to the Friedmann equation is just like pressureless matter (dark matter!).

The dark matter interpretation can be taken much further. Scalar fluctuation and CMB spectra. Consider now the fluctuating metrics

$$\begin{aligned} ds^2 &= -(1 + 2\Psi)dt^2 + a(t)^2(1 - 2\Phi)\delta_{ij}dx^i dx^j \\ df^2 &= -X(t)^2(1 + 2\Xi)dt^2 - Y^2\partial_i\beta dt dx^i \\ &\quad + Y(t)^2[(1 - 2\chi)\delta_{ij} + \partial_i\partial_j\mu]dx^i dx^j \end{aligned}$$

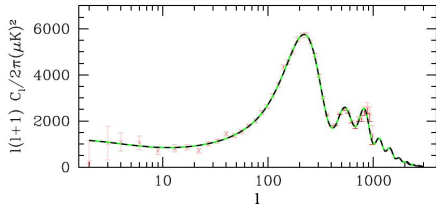
- ▶  $\Psi, \Phi$  are Newton potentials determined by Einstein equations.
- ▶  $\chi, \beta, \Xi, \mu$  are determined by the  $\frac{\delta I}{\delta f_{\mu\nu}} = 0$  equations and have a clear physical meaning. They represent a density contrast  $\delta$ , velocity fluctuation  $\Theta$ , shear  $\Pi$  and entropy  $S$ :

$$\begin{aligned} \delta &= \Psi - \Xi + 3(\Phi - \chi) \\ \Theta &= -\beta \\ \Pi &= w(\Xi - \Psi + \Phi - \chi) \\ S &= -w\mu \end{aligned}$$

Choosing  $\Pi = 0, S = 0$  as initial conditions the equations  $\frac{\delta I}{\delta f_{\mu\nu}} = 0$  imply

$$\dot{\delta} = -k^2\Theta + 3\dot{\Phi} \quad \dot{\Theta} = -\frac{\dot{a}}{a}\Theta + \Psi$$

which are exactly the correct equations for a dark matter fluid. Incorporating baryons, radiation, (no dark matter!) the CMB



$$\begin{aligned} &\langle \Delta T(\hat{n}_1) \Delta T(\hat{n}_2) \rangle \\ &= T_0^2 \sum_{\ell} \frac{C_{\ell}}{2\ell+1} P_{\ell}(\hat{n}_1 \cdot \hat{n}_2) \end{aligned}$$

(MB, P.Ferreira, C.Skordis, arXiv:0811.1272 for details.)

## Back to three dimensions. More applications

Cosmology: The evolution of the scale factor is wonderfully simple. A “simple” equation from the expansion factor  $a(t)$  can be deduced

$$\ddot{a} = \frac{\ddot{a}}{\dot{a}al} \left( -2al\ddot{a} + l\dot{a}^2 + \sqrt{4\dot{a}^2\dot{a}^2l^4\alpha + 4\ddot{a}a^3 - 2\dot{a}^2a^2} \right)$$

(We have eliminated  $f_{\mu\nu}$  hence  $\ddot{a}$ ). This equation has two de Sitter vacua: Setting  $a(t) = e^{Ht}$  there are two solutions for  $H$

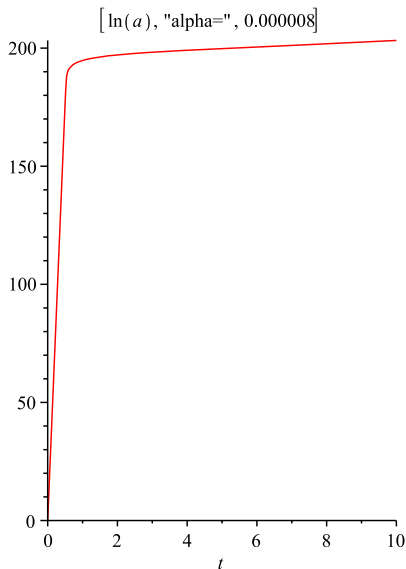
$$H_1 = \frac{1}{2\alpha l} \sqrt{2\alpha(1 + \sqrt{1 - 2\alpha})} \approx \frac{1}{l\sqrt{\alpha}} \left( 1 - \frac{\alpha}{4} + \dots \right)$$

$$H_2 = \frac{1}{2\alpha l} \sqrt{2\alpha(1 - \sqrt{1 - 2\alpha})} \approx \frac{\sqrt{2}}{2l} \left( 1 + \frac{\alpha}{4} + \dots \right)$$

Thus, if  $\alpha$  is small  $H_1$  can be arbitrary large while  $H_2$  is finite.

Number of  
e-foldings  $\sim 190$

The transition occur at  
 $\frac{t}{\ell} \sim 0.5$ .



Black holes: 3d bigravity also has black holes. The metric  $g_{\mu\nu}$  has exactly the same form as in pure general relativity:

$$ds^2 = -(-\Lambda_g r^2 - M)dt^2 + \frac{dr^2}{-\Lambda_g r^2 - M + \frac{J^2}{4r^2}} + Jdt d\phi + r^2 d\phi^2$$

$\Lambda_g$  is an arbitrary integration constant, as for Isham-Storey, which however will get fixed by asymptotic symmetries.

The metric  $f_{\mu\nu}$  is too long to be displayed. But it can be checked that both  $g_{\mu\nu}$  and  $f_{\mu\nu}$  have constant curvature (just like in NMG),

$$\begin{aligned} R(g)^{\mu\nu}{}_{\alpha\beta} &= \Lambda_g \delta_{[\alpha\beta]}^{[\mu\nu]} \\ R(f)^{\mu\nu}{}_{\alpha\beta} &= \Lambda_f \delta_{[\alpha\beta]}^{[\mu\nu]} \end{aligned}$$

Both black holes differ from AdS only in their global properties.

An interesting aspect of this solution is the fall off behavior of the  $f_{\mu\nu}$  black hole. Asymptotically, one has

$$f_{tr} \sim \frac{b}{r} + \mathcal{O}(1/r^3) \quad (8)$$

instead of

$$f_{tr} \sim \mathcal{O}(1/r^3).$$

as required by the Brown-Henneaux boundary conditions. This means that the  $f_{\mu\nu}$  field will contribute non-trivially to the asymptotic charges and will modify the CFT conformal symmetry.



CFT fields Using chiral coordinates  $z = t + \varphi$ ,  $\bar{z} = -t + \varphi$ , the following fields satisfy the asymptotic equations (T,Q,P, etc. are arbitrary functions of their arguments)

$$ds^2 \sim \frac{l^2 dr^2}{r^2} + r^2 dz d\bar{z} + T(z) dz^2 + \bar{T}(\bar{z}) d\bar{z}^2$$

$$df^2 \sim \frac{l^2 dr^2}{r^2} + r^2 dz d\bar{z} + Q(z) dz^2 + \bar{Q}(\bar{z}) d\bar{z}^2 + \frac{dr}{r} (P(z) dz + \bar{P}(\bar{z}) d\bar{z})$$

**The important point:** The fields  $P(z)$  and  $\bar{P}(\bar{z})$  can be set to zero by a coordinate redefinition. However, this field redefinition has already been done for the metric  $g_{\mu\nu}$  (Brown-Henneaux conditions). The freedom to make diffeomorphisms is exhausted to put the metric  $g_{\mu\nu}$  in its standard form. Then there is no more freedom for  $f_{\mu\nu}$  and the functions  $P(z)$  and  $\bar{P}(\bar{z})$  become physical.

Under Brown-Henneaux diffs (conformal transformation on the CFT),  $Q$  is a  $(2,0)$  field while  $P$  is  $(1,0)$ . They transform as

$$\delta Q = -\epsilon \partial Q - 2\partial\epsilon Q + P\partial^2\epsilon + \frac{1}{2}\partial^3\epsilon \quad (9)$$

$$\delta P = -\epsilon \partial P - \partial\epsilon P \quad (10)$$

The total generator of conformal transformations – conserved charge associated to the asymptotic symmetries – is

$$\hat{T}(z) = T(z) + Q(z) + \partial P(z)$$

$T(z)$  comes from the metric  $g_{\mu\nu}$ .  $Q + \partial P$  comes from  $f_{\mu\nu}$ . The total central charge is

$$c = \frac{3l}{3G} + \frac{3l}{2G}\sigma \quad (11)$$

- ▶ Two conformal points. We have displayed one conformal point where  $f_{tr} \sim 1/r^3$ . But there also exists solutions with  $f_{tr} \sim 1/r^3$  defining a different boundary CFT.
- ▶ The most important application to the Brown-Henneaux symmetry has been the microscopic counting of black hole degrees of freedom (Strominger, 1977). Does this also work in this case?

**Thank you**