Massive 3d gravity and Brown-Henneaux diffeomorphisms

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work (in progress) in collaboration with Stefan Theisen

TOC

- **TMG** (Topologically massive gravity) has received great attention recently. Li, Song and Strominger (2008) found a chiral point of TMG with interesting properties.
- **New Massive Gravity.** Bergshoeff, Holm and Townsend (2009) have constructed an extension of TMG for a massive graviton in three dimensions (omitting the Chern-Simons term):

$$I[g] = \int d^3x\sqrt{g}\left(\sigma R + \frac{1}{m^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\right) - 2\lambda m^2\right).$$

Despite being quadratic in the curvatures, this action has second order field equations. Linearization yields the Pauli-Fierz action for a massive spin-2 particle in three dimensions.

The trick: Add another field $f_{\mu\nu}$ and write

$$I[g,f] = \int \sqrt{g}\Big(R - 2\lambda m^2 + f^{\mu\nu}G_{\mu\nu} - \frac{m^2}{4}(f^{\mu\nu}f_{\mu\nu} - f^2)\Big)$$

$f_{\mu\nu}$ is non-dynamical and the actions are the same. Linearizing

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + ah_{\mu\nu} + bk_{\mu\nu}, \qquad f_{\mu\nu} = \bar{g}_{\mu\nu} + ch_{\mu\nu} + c\,k_{\mu\nu}.$$

a, b, c, d can be chosen such that the fluctuations $h_{\mu\nu}$ and $k_{\mu\nu}$ are decoupled. The quadratic action is

$$I[h,k] = \int \underbrace{h^{\mu\nu}\mathcal{G}(h)_{\mu\nu}}_{\text{(trivial in 3d)}} \quad + \quad \underbrace{k^{\mu\nu}\mathcal{G}(k)_{\mu\nu} - \frac{m^2}{4}(k^{\mu\nu}k_{\mu\nu} - k^2)}_{\text{(massive spin-2 particle)}}$$

$$h^{\mu\nu}\mathcal{G}(h)_{\mu\nu} \equiv \frac{-1}{4}D_\mu h_{\nu\rho}D^\mu h^{\nu\rho} + \frac{1}{2}D_\lambda h_{\mu\nu}D^\mu h^{\lambda\nu} + \frac{1}{4}D_\mu h D^\mu h$$
$$-\frac{1}{2}D_\mu h D_\nu h^{\mu\nu} + \frac{1}{2l^2}\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right)$$

- **'Old-New' massive gravity**. A similar structure appears in a different way, proposed almost 40 years (!) ago by Isham, Salam and Strathdee. Consider an action for *two* spin-2 fields

$$
\begin{aligned}
I[g_{\mu\nu}, f_{\mu\nu}] \;\; &= \;\; \frac{1}{16\pi G}\int \sqrt{g}(R_g - \Lambda) + \sigma\sqrt{f}(R_f - \Lambda) + \\
&\quad m^2\sqrt{g}(g_{\mu\nu} - f_{\mu\nu})(g_{\alpha\beta} - f_{\alpha\beta})(f^{\mu\alpha}f^{\nu\beta} - f^{\mu\nu}f^{\alpha\beta})
\end{aligned}
$$

Again the action for fluctuations can be diagonalized

$$
g_{\mu\nu} = \bar{g}_{\mu\nu} + ah_{\mu\nu} + bk_{\mu\nu}, \qquad f_{\mu\nu} = \bar{g}_{\mu\nu} + ch_{\mu\nu} + d\,k_{\mu\nu}.
$$

obtaining the same linear action as Bergshoeff, Hohm, Townsend

$$
I[h,k] = \int h^{\mu\nu}\mathcal{G}(h)_{\mu\nu} \quad + \quad k^{\mu\nu}\mathcal{G}(k)_{\mu\nu} - \frac{m^2}{4}(k^{\mu\nu}k_{\mu\nu} - k^2)
$$

(trivial in 3d) (massive spin-2 particle)

Why multigravity? Maxwell theory is extended into Yang-Mills theory:

$$\int F_{\mu\nu}F^{\mu\nu} \quad \rightarrow \quad \int F_{1\,\mu\nu}F_1^{\mu\nu} + F_{2\,\mu\nu}F_2^{\mu\nu} + ... + \textbf{interactions}$$

This extension has deep applications (in Physics and Mathematics). It is extremely tempting to do the same with the spin two field:

$$\int \sqrt{g}(R - 2\Lambda) \quad \rightarrow \quad \int \sqrt{g_1}(R_1 - 2\Lambda_1) + \sqrt{g_2}(R_2 - 2\Lambda_2) + ... \\ + \textbf{interactions}$$

However, for spin 2, the interactions do not have the same nice structure:

The N-Gravity action functional

$$I[g^a_{\mu\nu}] = \sum_{a=1}^{N} \int \sqrt{g_a} R_a \tag{1}$$

is invariant under independent transformations on each metric

$$\delta g^a_{\mu\nu} = \xi^\alpha_{a\,,\mu} g^a_{\alpha\nu} + \xi^\alpha_{a\,,\nu} g^a_{\mu\alpha} + \xi^\alpha_a g^a_{\mu\nu,\alpha}, \qquad a = 1...N. \tag{2}$$

Is this symmetry (or a deformation of it) preserved by interactions? For example,

$$g_1^{\mu\nu} g_{2\,\mu\nu} \tag{3}$$

is invariant *provided* $\xi_1^\mu = \xi_2^\mu$. A general perturbative theorem has been proved. For weak fields,

$$g^a_{\mu\nu} = \eta_{\mu\nu} + h^a_{\mu\nu} \tag{4}$$

no cross potential $V(h^a_{\mu\nu})$ preserving the full N dimensional symmetry exists (Boulanger-Damour-Gualtieri-Henneaux (2000)).

Interactions of the Isham, Salam, Strathdee family $(u+v=\frac{1}{2})$

$$V = \det(g)^u \det(g)^v \left(f^{\mu\nu} g_{\mu\nu} + \kappa((f^{\mu\nu} g_{\mu\nu})^2 - f^{\mu\nu} g_{\nu\alpha} f^{\alpha\beta} g_{\beta\mu}) \right)$$

preserve the diagonal subgroup.

- If there are two metrics, different matter fields may couple to different metrics. This implies that while standard particles follow geodesics of $g_{\mu\nu}$, other exotic particles may follow geodesics of $f_{\mu\nu}$

$$\frac{d^2 y^\mu}{ds^2} + \Gamma(f)^\mu{}_{\alpha\beta} \frac{dy^\alpha}{ds} \frac{dy^\beta}{ds} = 0$$

  Wouldn't it be interesting?

- See Damour,Kogan; Arkani-Hamed,Georgi,Schwartz; Blas,Deffayet,Garriga; Berezhiani, Comelli, Nesti, Pilo; Blas,Comelli,Nesti,Pilo;... for more work on this.

Isham, Storey (1978) found the exact Schwarzschild (type II) metric for the Isham-Salam-Strathdee potential,

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{r}+\Lambda_1 r^2\right)^2 dt^2 + \frac{dr^2}{1-\frac{2M}{r}+\Lambda_1 r^2} + r^2 d\Omega^2 \\
df^2 &= -\left(1-\frac{2\mu}{r}+\Lambda_2 r^2\right)^2 dt'^{\,2} + \frac{dr^2}{1-\frac{2\mu}{r}+\Lambda_2 r^2} + r^2 d\Omega^2
\end{aligned}
$$

where

$$t' = t + \text{`function'}(r)$$

- Λ_1 and Λ_2 are integration constants not related to action couplings. Asymptotically AdS spaces coexist with de-Sitter cosmologies as solutions to the same action!
- Thermodynamics? Charges? (are finite); Entropy? To be done.

Cosmology: The Friedmann equation for the ansatz

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 d\vec{x}^2, &(5)\\ df^2 &= -X(t)^2 dt^2 + Y(t)^2 d\vec{x}^2 &(6) \end{aligned}$$

is

$$\frac{\dot{a}^2}{a^2} = \frac{Y^3/X}{a^3} + \frac{\Omega_{baryons}}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda \qquad (7)$$

A remarkable property of this class of theories is that near $a = 0$ Y^3/X is constant. Thus we can set

$$\frac{Y^3}{X} = \Omega_{dark\ matter}$$

and the contribution from $f_{\mu\nu}$ to the Friedmann equation is just like pressureless matter (dark matter!).

The dark matter interpretation can be taken much further. Scalar fluctuation and CMB spectra. Consider now the fluctuating metrics

$$
\begin{aligned}
ds^2 &= -(1+2\Psi)dt^2 + a(t)^2(1-2\Phi)\delta_{ij}dx^i dx^j \\
df^2 &= -X(t)^2(1+2\Xi)dt^2 - Y^2\partial_i\beta\, dtdx^i \\
&\quad + Y(t)^2[(1-2\chi)\delta_{ij} + \partial_i\partial_j\mu]dx^i dx^j
\end{aligned}
$$

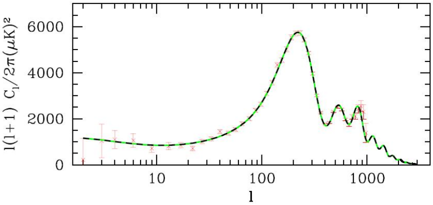
- Ψ, Φ are Newton potentials determined by Einstein equations.
- χ, β, Ξ, μ are determined by the $\frac{\delta I}{\delta f_{\mu\nu}} = 0$ equations and have a clear physical meaning. They represent a density contrast δ, velocity fluctuation Θ, shear Π and entropy S:

$$
\begin{aligned}
\delta &= \Psi - \Xi + 3(\Phi - \chi) \\
\Theta &= -\beta \\
\Pi &= w(\Xi - \Psi + \Phi - \chi) \\
S &= -w\,\mu
\end{aligned}
$$

Choosing $\Pi = 0, S = 0$ as initial conditions the equations $\frac{\delta I}{\delta f_{\mu\nu}} = 0$ imply

$$\dot{\delta} = -k^2\Theta + 3\dot{\Phi} \qquad \dot{\Theta} = -\frac{\dot{a}}{a}\Theta + \Psi$$

which are exactly the correct equations for a dark matter fluid. Incorporating baryons, radiation, (no dark matter!) the CMB



$$\langle \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\rangle$$

$$= T_0^2 \sum_\ell \frac{C_\ell}{2\ell+1} P_\ell(\hat{n}_1 \cdot \hat{n}_2)$$

(MB, P.Ferreira, C.Skordis, arXiv:0811.1272 for details.)

Back to three dimensions. More applications

Cosmology: The evolution of the scale factor is wonderfully simple. A "simple" equation form the expansion factor $a(t)$ can be deduced

$$\dddot{a} = \frac{\ddot{a}}{\dot{a}a\ell}\left(-2a\ell\ddot{a} + \ell\dot{a}^2 + \sqrt{4\ddot{a}^2\dot{a}^2\ell^4\alpha + 4\ddot{a}a^3 - 2\dot{a}^2a^2}\right)$$

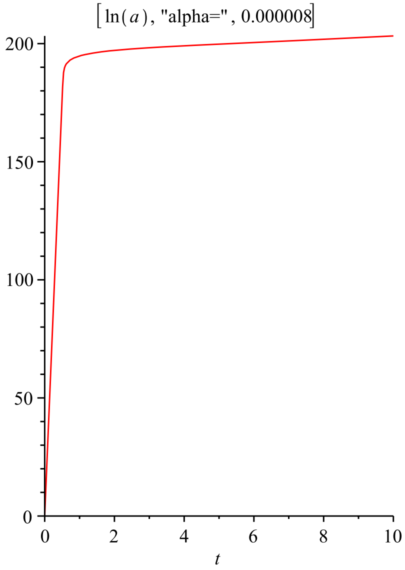
(We have eliminated $f_{\mu\nu}$ hence $\dddot{a}$.). This equation has two de Sitter vacua: Setting $a(t) = e^{Ht}$ there are two solutions for H

$$\begin{aligned} H_1 &= \frac{1}{2\alpha\ell}\sqrt{2\alpha(1+\sqrt{1-2\alpha})} \approx \frac{1}{\ell\sqrt{\alpha}}\left(1-\frac{\alpha}{4}+\cdots\right) \\ H_2 &= \frac{1}{2\alpha\ell}\sqrt{2\alpha(1-\sqrt{1-2\alpha})} \approx \frac{\sqrt{2}}{2\ell}\left(1+\frac{\alpha}{4}+\cdots\right) \end{aligned}$$

Thus, if α is small H_1 can be arbitrary large while H_2 is finite.

Number of e-foldings ~ 190

The transition occur at $\frac{t}{\ell} \sim 0.5$.



Black holes: 3d bigravity also has black holes. The metric $g_{\mu\nu}$ has exactly the same form as in pure general relativity:

$$ds^2 = -(-\Lambda_g r^2 - M)dt^2 + \frac{dr^2}{-\Lambda_g r^2 - M + \frac{J^2}{4r^2}} + Jdtd\phi + r^2 d\phi^2$$

Λ_g is an arbitrary integration constant, as for Isham-Storey, which however will get fixed by asymptotic symmetries.

The metric $f_{\mu\nu}$ is too long to be displayed. But it can be checked that both $g_{\mu\nu}$ and $f_{\mu\nu}$ have constant curvature (just like in NMG),

$$\begin{aligned} R(g)^{\mu\nu}{}_{\alpha\beta} &= \Lambda_g \delta^{[\mu\nu]}_{[\alpha\beta]} \\ R(f)^{\mu\nu}{}_{\alpha\beta} &= \Lambda_f \delta^{[\mu\nu]}_{[\alpha\beta]} \end{aligned}$$

Both black holes differ from AdS only in their global properties.

An interesting aspect of this solution is the fall off behavior of the $f_{\mu\nu}$ black hole. Asymptotically, one has

$$f_{tr} \sim \frac{b}{r} + \mathcal{O}(1/r^3) \tag{8}$$

instead of

$$f_{tr} \sim \mathcal{O}(1/r^3).$$

as required by the Brown-Henneaux boundary conditions. This means that the $f_{\mu\nu}$ field will contribute non-trivially to the asymptotic charges and will modify the CFT conformal symmetry.

CFT fields Using chiral coordinates $z = t + \varphi, \bar{z} = -t + \varphi$, the following fields satisfy the asymptotic equations (T,Q,P, etc. are arbitrary functions of their arguments)

$$
\begin{aligned}
ds^2 &\sim \frac{l^2 dr^2}{r^2} + r^2 dz d\bar{z} + T(z) dz^2 + \bar{T}(\bar{z}) dz^2 \\
df^2 &\sim \frac{l^2 dr^2}{r^2} + r^2 dz d\bar{z} + Q(z) dz^2 + \bar{Q}(\bar{z}) dz^2 + \frac{dr}{r}(P(z) dz + \bar{P}(\bar{z}) d\bar{z}
\end{aligned}
$$

The important point: The fields $P(z)$ and $\bar{P}(\bar{z})$ can be set to zero by a coordinate redefinition. However, this field redefinition has already been done for the metric $g_{\mu\nu}$ (Brown-Henneaux conditions). The freedom to make diffeomorphisms is exhausted to put the metric $g_{\mu\nu}$ in its standard form. Then there is no more freedom for $f_{\mu\nu}$ and the functions $P(z)$ and $\bar{P}(\bar{z})$ become physical.

Under Brown-Henneaux diffs (conformal transformation on the CFT), Q is a $(2,0)$ field while P is (1,0). They transforms as

$$\delta Q = -\epsilon\partial Q - 2\partial\epsilon Q + P\partial^2\epsilon + \frac{1}{2}\partial^3\epsilon \tag{9}$$
$$\delta P = -\epsilon\partial P - \partial\epsilon P \tag{10}$$

The total generator of conformal transformations –conserved charge associated to the asymptotic symmetries– is

$$\hat{T}(z) = T(z) \quad + \quad Q(z) + \partial P(z)$$

$T(z)$ comes from the metric $g_{\mu\nu}$. $Q + \partial P$ comes from $f_{\mu\nu}$. The total central charge is

$$c = \frac{3l}{3G} \; + \; \frac{3l}{2G}\sigma \tag{11}$$

- Two conformal points. We have displayed one conformal point where $f_{tr} \sim 1/r^3$. But there also exists solutions with $f_{tr} \sim 1/r^3$ defining a different boundary CFT.
- The most important application to the Brown-Henneaux symmetry has been the microscopic counting of black hole degrees of freedom (Strominger, 1977). Does this also work in this case?

Thank you