Causality and Statistics on the Moyal Plane

Wroclaw

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1. WHAT IS CAUSALITY

There are different, apparently unrelated, concepts.

KRAMERS-KRONIG RELATION

The system should not respond before the time at which it is disturbed. If $R(t)$ is response and disturbance of system is zero for time < 0,

$$R(t) = 0 \quad t < 0. \tag{1}$$

Its Fourier transform

$$\tilde{R}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega\, t} R(t) = \int_{0}^{\infty} e^{i\omega\, t} R(t) \tag{2}$$

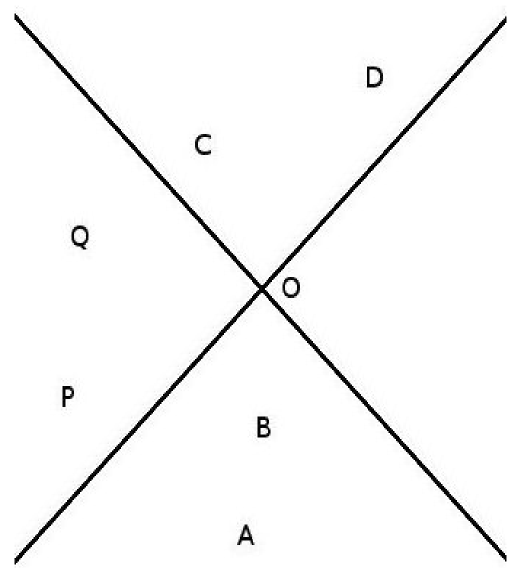
is holomorphic in

$$\Im \omega > 0. \tag{3}$$

SORKIN

Causality for events is a partial order $>$:

"$A > B$ if A is to future of B."



O is the observer. C, D are in the future, and A, B are in the past light cone of O. They are causally related to O. Q, P are spacelike relative to O and are not causally related to O.

Observables $\rho(x)$, $\eta(y)$ commute if x and y are spacelike separated:

$$[\rho(x), \eta(y)] = 0 \tag{4}$$

if

$$(x^0 - y^0)^2 - (\vec{x} - \vec{y})^2 < 0 \quad \text{or} \quad x \sim y. \tag{5}$$

Enforced by

$$[\varphi(x), \chi(y)]_- = 0 \quad x \sim y \tag{6}$$

for scalar fields,

$$\left[\psi_\alpha^{(1)}(x), \psi_\beta^{(2)}(y)\right]_+ = 0 \quad x \sim y \tag{7}$$

for spinor fields.

But these also express Statistics.

So Causality and Statistics are connected.

Moyal Plane describes noncommutative spacetime where commutation relations and hence causality and statistics are deformed.

2. THE MOYAL PLANE

The Groenewold - Moyal (G-M) plane is the algebra of functions $\mathcal{A}_\theta$ on $\mathbb{R}^{d+1}$ with a twisted product:

$$f * g(x) = f\, e^{i/2 \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu}\, g.$$

It implies that spacetime is noncommutative:

$$\widehat{x}_\mu \star \widehat{x}_\nu - \widehat{x}_\nu \star \widehat{x}_\mu = \left[\widehat{x}_\mu, \widehat{x}_\nu\right]_\star = i\theta_{\mu\nu}, \quad \mu, \nu = 0, 1, \cdots, d.$$

$$\widehat{x}_\mu = \text{coordinate functions}, \quad \widehat{x}_\mu(x) = x_\mu.$$

In this talk I will describe a particular approach to the formulation of quantum field theories on the GM plane and indicate its physical consequences.

3. RELEVANCE IN PHYSICS

The following arguments were described by Doplicher, Fredenhagen and Roberts.

SPACE-SPACE NONCOMMUTATIVITY

In order to probe physics at the Planck scale L, the Compton wavelength $\frac{\hbar}{Mc}$ of the probe must fulfill

$$\frac{\hbar}{Mc} \leq L \;\; or \;\; M \geq \frac{\hbar}{Lc} \simeq \text{Planck mass.}$$

Such high mass in the small volume L^3 will strongly affect gravity and can cause black holes and their horizons to form. This suggests a fundamental length limiting spatial localization.

TIME-SPACE NONCOMMUTATIVITY

Similar arguments can be made about time localization.

Observation of very short time scales requires very high energies. They can produce black holes and black hole horizons will then limit spatial resolution suggesting

$$\Delta t \, \Delta|\overrightarrow{x}| \geq L^2 , \quad L = \text{a fundamental length.}$$

The G-M plane *models* above spacetime uncertainties.

4. THE TWISTED COPRODUCT

If there is a symmetry group G with elements g and it acts on single particle Hilbert spaces $\mathcal{H}_i$ by unitary representations $g \to U_i(g)$, then conventionally it acts on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by the representation

$$g \to [U_1 \otimes U_2](g \otimes g).$$

The homomorphism

$$\begin{aligned} \Delta : G &\to G \otimes G, \\ g &\to \Delta(g) := g \otimes g \end{aligned}$$

underlying these equations is said to be a coproduct on G.

The action of G on multiparticle states involves more than just group theory. It involves the coproduct Δ.

Let

$$F_\theta = e^{\frac{i}{2}\partial_\mu \otimes \theta^{\mu\nu}\partial_\nu} = \text{“Twist element”}.$$

Then

$$f * g = m_0[F_\theta f \otimes g]$$

where m_0 is the point-wise multiplication map of $\mathcal{A}_0$:

$$m_0(\alpha \otimes \beta)(x) = \alpha(x)\beta(x)$$

Let Λ be an element of the Poincaré group $\mathcal{P}_+^\uparrow$. For $x \in \mathbb{R}^N$,

$$\Lambda : x \to \Lambda x \in \mathbb{R}^N.$$

It acts on functions on $\mathbb{R}^N$ by pull-back:

$$\Lambda : \alpha \to \Lambda^* \alpha, \quad (\Lambda^* \alpha)(x) = \alpha[\Lambda^{-1} x].$$

The work of Aschieri et al. and Chaichian et al. based on Drinfel'd's basic paper shows that $\mathcal{P}_+^\uparrow$ acts on $\mathcal{A}_\theta(\mathbb{R}^N)$ compatibly with m_θ if its coproduct is "twisted" to Δ_θ where

$$\Delta_\theta(\Lambda) = F_\theta^{-1} (\Lambda \otimes \Lambda) F_\theta.$$

5. THE TWISTED STATISTICS

Twisting the coproduct implies twisting of statistics in quantum theory.

QUANTUM MECHANICS

A two-particle system for $\theta^{\mu\nu} = 0$ is a function of two sets variables, lives in $\mathcal{A}_0 \otimes \mathcal{A}_0$. It transforms according to the usual coproduct Δ_0.

Similarly in noncommutative case, the wavefunction lives in $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ and transforms according to the twisted coproduct Δ_θ.

For $\theta^{\mu\nu} = 0$ we require that the physical wave functions describing identical particles are either symmetric (bosons) or antisymmetric (fermions).

That is we work with either the symmetrized or antisymmetrized tensor product

$$\phi \otimes_{S,A} \chi \equiv \frac{1}{2}(\phi \otimes \chi \pm \chi \otimes \phi)$$

In a Lorentz-invariant theory, these relations have to hold in all frames of reference.

The twisted coproduct action of the Lorentz group is not compatible with the usual symmetrization/antisymmetrization.

Thus let τ_0 be the statistics (flip) operator associated with exchange for $\theta^{\mu\nu} = 0$:

$$\tau_0(\phi \otimes \chi) = \chi \otimes \phi \, .$$

For $\theta^{\mu\nu} = 0$, we have the axiom that τ_0 is superselected. In particular, for Lorentz group action, $\Delta_0(\Lambda) = \Lambda \otimes \Lambda$, must and *does* commute with the statistics operator:

$$\tau_0 \, \Delta_0(\Lambda) = \Delta_0(\Lambda)\tau_0.$$

Given an element $\phi \otimes \chi$ of the tensor product, the physical Hilbert spaces can be constructed from the elements

$$\left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi) .$$

Now

$$\tau_0 F_\theta = F_\theta^{-1} \tau_0$$

so that

$$\tau_0 \, \Delta_\theta(\Lambda) \neq \, \Delta_\theta(\Lambda) \tau_0$$

showing that the usual statistics is not compatible with the twisted co-product.

But the new statistics operator

$$\tau_\theta \equiv F_\theta^{-1} \tau_0 F_\theta, \quad \tau_\theta^2 = \mathbf{1} \otimes \mathbf{1}$$

does commute with the twisted coproduct Δ_θ:

$$\Delta_\theta(\Lambda) = F_\theta^{-1} \Lambda \otimes \Lambda \, F_\theta.$$

The states constructed according to

$$\phi \otimes_{S_\theta} \chi \equiv \left(\frac{1 + \tau_\theta}{2} \right) (\phi \otimes \chi),$$

$$\phi \otimes_{A_\theta} \chi \equiv \left(\frac{1 - \tau_\theta}{2} \right) (\phi \otimes \chi).$$

form the physical two-particle Hilbert spaces of (generalized) bosons and fermions and obey twisted statistics.

6. THE PAULI PRINCIPLE

In an interesting paper, (hep-th/0601121, Twisted Galilean symmetry and the Pauli principle at low energies, Authors: Biswajit Chakraborty, Sunandan Gangopadhyay, Arindam Ghosh Hazra, Frederik. G. Scholtz), the statistical potential V_{STAT} between fermions has been computed:

$$\exp\Big(-\beta V_{\mathrm{STAT}}(\overrightarrow{\mathbf{x}}_1, \overrightarrow{\mathbf{x}}_2)\Big) = \langle \overrightarrow{\mathbf{x}}_1, \overrightarrow{\mathbf{x}}_2 | e^{-\beta H} | \overrightarrow{\mathbf{x}}_1, \overrightarrow{\mathbf{x}}_2 \rangle,$$

$$H = \frac{1}{2m}(\overrightarrow{\mathbf{p}}_1^2 + \overrightarrow{\mathbf{p}}_2^2).$$

Here $|\overrightarrow{\mathbf{x}}_1, \overrightarrow{\mathbf{x}}_2\rangle$ has twisted antisymmetry:

$$\tau_\theta|\overrightarrow{\mathbf{x}}_1, \overrightarrow{\mathbf{x}}_2\rangle = -|\overrightarrow{\mathbf{x}}_1, \overrightarrow{\mathbf{x}}_2\rangle.$$

It is explicitly shown not to have an infinitely repulsive core, establishing the violation of Pauli principle, as we had earlier suggested.

This result has phenomenological consequences.

It affects the Chandrasekhar limit on stars.It will also predict Pauli forbidden transitions on which there are stringent limits.

For example, in the Borexino and Super Kamiokande experiments, the forbidden transitions from O^{16}(C^{12}) to $\tilde{O}^{16}$($\tilde{C}^{12}$) where the tilde nuclei have an extra nucleon in the filled $1S_{1/2}$ level are found to have lifetimes greater than 10^{27} years.

There are also experiments on forbidden transitions to filled K-shells of crystals done in Maryland which give branching ratios less than 10^{-25} for such transitions.The consequences of these results to noncommutative models are yet to be studied.

7. COSMIC MICROWAVE BACKGROUND (CMB)

In 1992, the COBE satellite detected anisotropies in the CMB radiation, which led to the conclusion that the early universe was not smooth; there were small perturbations in the photon-baryon fluid.

The perturbations could be due to the quantum fluctuations in the inflaton (the scalar field driving inflation).

The quantum fluctuations were transmitted into the metric as primordial perturbations.

The temperature field in the sky can be expanded in spherical harmonics:

$$\frac{\Delta T(\hat{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\hat{n}).$$

The a_{lm} can be written in terms of perturbations to Newtonian potential Φ

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \Phi(k) \Delta_l^T(k) Y_{lm}^*(\hat{k})$$

where $\Delta_l^T(k)$ are called the transfer functions.

We find

$$\begin{aligned} \langle a_{lm} a_{l'm'}^* \rangle_\theta &= 8\pi^2 \int dk \sum_{l''=0,\ l'':\text{even}}^{\infty} i^{l+l'} (-1)^{l+m} (2l''+1)\, k^2 \Delta_l(k) \Delta_{l'}(k) P_\Phi(k) i_{l''}(\theta k H) \\ & \times \sqrt{(2l+1)(2l'+1)} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & 0 \end{pmatrix} \end{aligned}$$

where $\vec{\theta}^0 = \theta(0,\ 0,\ 1)$.

P_Φ is the power spectrum

$$P_\varphi = \frac{8\pi G}{9\varepsilon}\frac{H^2}{k^3}\Big|_{aH=k}$$

H is the Hubble parameter and ε is the slow-roll parameter.

The angular correlator $\langle a_{lm} a^*_{l'm'} \rangle_\theta$ is θ dependent indicating a preferred direction.

Correlation functions are not invariant under rotations. They are not Gaussian either.

It clearly breaks isotropy in the CMB radiation.

On fitting data, one finds,

$\sqrt{\theta} \lesssim 10^{-17}$ cms or energy scale $\gtrsim 10^4$ GeV.

8. CAUSALITY AND LORENTZ INVARIANCE

Lorentz Noninvariance:

The S-matrix is not Lorentz invariant.The reason is loss of causality:

Let H_I be the interaction Hamiltonian density in the interaction representation. The interaction representation S-matrix is

$$S = T \exp\left(-i \int d^4 x H_I(x)\right).$$

Bogoliubov and Shirkov and then Weinberg long ago deduced from causality (locality) and relativistic invariance that H_I must a local field:

$$[H_I(x), H_I(y)] = 0, x \sim y$$

where $x \sim y$ means x and y are space-like separated.

But noncommutative theories are nonlocal and violate this condition: this is the essential reason for Lorentz noninvariace.

The effect on scattering amplitudes is striking. They depend on total incident momentum $\overrightarrow{P}_{\text{inc}}$ through

$$\theta_{0i}\left(\overrightarrow{P}_{\text{inc}}\right)_i$$

So effects of $\theta_{\mu\nu}$ disappear in the center-of-mass system, or more generally if

$$\theta_{0i}\left(\overrightarrow{P}_{\rm inc}\right)_i = 0$$

But otherwise there is dependence on θ_{0i}.

The violation is of order $\theta_{0i}\left(\overrightarrow{P}_{\rm inc}\right)_i$ in cross sections.

REMARK: Even with noncommutativity $Z^0 \longrightarrow 2\gamma$ is forbidden in the approach of Aschieri et. al.

More generally,

A massive particle of spin j does not decay into two massless particles of same helicity if j is odd.

CPT :

The effect of P and CPT is to reverse the sign of θ_{0i}:

$$\text{P or CPT} : \theta_{0i} \to -\theta_{0i}.$$

The θ_{0i} contributes to P, and more strikingly, to CPT violation.

Particle-antiparticle life times can differ to order θ_{0i}:

$$\tau_{\text{particle}} - \tau_{\text{antiparticle}} \cong \theta_{0i} \left(\vec{P}_{\text{inc}} \right)_i.$$

$(g-2)$ of μ^-, μ^+ can differ.

We are estimating bounds on θ from these effects.

9. FINAL REMARKS

Noncommutative spacetimes deform statistics and so generically violate causality.
Such violations lead to

- correlations of observables in spacelike regions
- applications to homogeneity problem in cosmology?
- modification of Pauli principle
- Lorentz and CPT violations in scattering amplitudes.

There are specific predictions that may be observable.